

AUTHOR: Leobardo Rosales

TITLE: Partial boundary regularity for co-dimension one area-minimizing currents at immersed $C^{1,\alpha}$ tangential boundary points.

ABSTRACT: We give partial boundary regularity for co-dimension one absolutely area-minimizing currents at points where the boundary consists of a sum of $C^{1,\alpha}$ submanifolds, possibly with multiplicity, meeting tangentially, given that the current has a tangent cone supported in a hyperplane with constant orientation vector; this partial regularity is such that we can conclude the tangent cone is unique. The proof closely follows that giving the boundary regularity result of Hardt and Simon in [14].

KEYWORDS: Currents; Area-minimizing; Boundary Regularity.

MSC numbers: 28A75; 49Q05; 49Q15;

1 Introduction

Through a careful modification of the work found in [14], we are able to give partial regularity for co-dimension one absolutely area-minimizing currents at points where the boundary is tangentially $C^{1,\alpha}$ immersed. Our main result, Theorem 2, can be heuristically stated as follows:

Theorem 2 *Suppose T is an n -dimensional absolutely area-minimizing integer rectifiable current in an open subset of \mathbb{R}^{n+1} containing the origin, and that near the origin ∂T consists of a sum of $(n-1)$ -dimensional $C^{1,\alpha}$ orientable submanifolds for some $\alpha \in (0, 1]$, each possibly with multiplicity, meeting tangentially (with same orientation) at the origin. Suppose as well that T has a tangent cone at the origin*

$$\mathbb{C} = M\mathbb{E}^n \llcorner \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_n > 0\} + m\mathbb{E}^n \llcorner \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_n < 0\}$$

where M, m are positive integers with $m \leq M-1$. Then near the origin, there is a large region of the horizontal hyperplane $\mathbb{R}^n \times \{0\}$ such that the support of T over this region is the graph of a $C^{1, \frac{\alpha}{4n+6}}$ function.

Furthermore, the region is such that we can conclude \mathbb{C} is the unique tangent cone of T at the origin. Here, \mathbb{E}^n is the current associated to the hyperplane $\mathbb{R}^n \times \{0\}$ with usual orientation; see 4.1.7 of [10]. See 4.3.16 of [10] for the definition of a tangent cone of a current.

Theorem 2 is precisely a generalization of Corollary 9.3 of [14], after applying the Hopf-type boundary point lemma given by Lemma 7 of [11], also appearing in [14] as Lemma 10.1. We can get full boundary regularity via [30] in the special case that ∂T is supported on exactly one $C^{1,\alpha}$ submanifold (if for example $m = M - 1$), letting in this case $m \in \{0, \dots, M - 1\}$ and $M \geq 1$. By [30] and the fact that the tangent cone of T at the origin is \mathbb{C} as above, if $m = 0$ then T corresponds to a $C^{1,\alpha}$ hypersurface-with-boundary, and if $m \geq 1$ then the support of T near the origin is a real analytic hypersurface, with T having multiplicity M, m on either side of ∂T .

1.1 Modifying the work of Hardt and Simon

To the reader thoroughly familiar with the entirety of [14], we can describe the key estimate and minor modification which allows us to carry over the proofs found in [14].

First, the key estimate needed is given by (A.0.8). This estimate is crucial to prove Lemma 8.4, analogous to Lemma 6.4 of [14]. We use (A.0.8) to prove Lemma 8.4, to show the function defined by taking the top sheet of the harmonic blowups (when the harmonic blowups are linear as in Lemma 8.4) is harmonic, leading to conclude the harmonic blowups are all given by the same linear function. See (8.4.15), where we specifically refer to (A.0.8) in the proof of Lemma 8.4.

The minor modification which must be made is seen in Lemma 5.2, which is analogous to Lemma 3.2 of [14]. One can see in the right-hand side of the conclusion, we have replaced $c_7 \tau^{-2} \kappa_T^2$ in Lemma 3.2 of [14] with $c_7 \kappa_T$ in Lemma 5.2. This difference arises from the fact that, as opposed to [14], the boundary ∂T no longer corresponds to integrating over an embedded submanifold. As such, a slightly different proof must be given for Lemma 5.2 than the proof of Lemma 3.2 of [14]. We must subsequently take care that the rest of [14] follows through keeping in mind Lemma 5.2; notably, we must check that Lemmas 6.1, 8.4 still hold.

Besides these two points, the rest of [14] passes through essentially without change, with only minor modifications due to the more general structure of ∂T . For the reader already familiar with [14] who wishes a more terse exposition, we reference [22], which is a shorter version of this work in which only the differences with [14] are explained. However, we take this

opportunity to reintroduce the seminal work of [14], using more modern notation. We also include clarifying exposition, some of which is taken from [7], which extends the results of [14] to minimizing currents with prescribed mean curvature.

1.2 An application of Theorem 2

We note an application of Theorem 2, which in fact motivated the present work. Recently in [21] the author introduced the c -isoperimetric mass of currents, which is defined for each $c > 0$ by

$$\mathbf{M}^c(T) = \mathbf{M}(T) + c\mathbf{M}(\partial T)^\kappa$$

whenever T is an n -dimensional integer multiplicity rectifiable current in \mathbb{R}^{n+k} , \mathbf{M} is the usual mass on currents, and $\kappa = \frac{n}{n-1}$ is the isoperimetric exponent.

This leads to define and study a minimization problem. Let Γ be an $(n-1)$ -dimensional integer rectifiable current in \mathbb{R}^{n+k} with compact support and $\partial\Gamma = 0$, which we refer to as the fixed boundary. Define $\mathbf{I}_\Gamma(\mathbb{R}^{n+k})$ to be the set of n -dimensional integer rectifiable currents T with compact support so that $\partial T = \Gamma + \Sigma$ where Γ and Σ have disjoint supports. We then say $\mathbf{T}_c \in \mathbf{I}_\Gamma(\mathbb{R}^{n+k})$ is a solution to the c -Plateau problem with respect to fixed boundary Γ if \mathbf{T}_c minimizes \mathbf{M}^c amongst all $T \in \mathbf{I}_\Gamma(\mathbb{R}^{n+k})$ (see Definition 3.3 of [21] with $U = \mathbb{R}^{n+k}$). For such \mathbf{T}_c , writing $\partial\mathbf{T}_c = \Gamma + \Sigma_c$ we refer to Σ_c as the free boundary.

Theorem 8.2 of [21] concludes there is no solution to the c -Plateau problem \mathbf{T}_c with $\partial\mathbf{T}_c = \Gamma + \Sigma_c$ with nonzero free boundary Σ_c a smooth embedded $(n-1)$ -dimensional submanifold with parallel mean curvature (that is constant mean curvature in the sense of [15]) so that \mathbf{T}_c near Σ_c is a smooth submanifold-with-boundary. This can be used in Theorem 9.1 of [21] to show that in case the fixed boundary Γ is one-dimensional in the plane, that is if $n = 2, k = 0$, then free boundaries must always be empty. However, so-called non-trivial solutions in the limit can occur, as seen in Theorem 10.2 of [21] which shows that for small values of $c > 0$ when Γ is the square in the plane, the infimum of \mathbf{M}^c is attained in the limit by a sequence of currents in $\mathbf{I}_\Gamma(\mathbb{R}^2)$ which converge to a nonempty current not in $\mathbf{I}_\Gamma(\mathbb{R}^2)$.

The author conjectures that this holds generally in $n = 2, k = 1$: if the fixed boundary Γ is one-dimensional in \mathbb{R}^3 , then for each $c > 0$ either every

solution to the c -Plateau problem \mathbf{T}_c with fixed boundary Γ has empty free boundary, so that $\partial\mathbf{T}_c = \Gamma$, or the infimum value of \mathbf{M}^c can only be attained in the limit by a sequence of currents in $\mathbf{I}_\Gamma(\mathbb{R}^3)$. Evidence for this is given by recent work by the author in [23], where it is proved (in case $n = 2, k = 1$) that \mathbf{T}_c at singular points of the free boundary Σ_c must have complicated topology; more specifically, \mathbf{T}_c cannot be supported in a finite union of C^1 surfaces-with-boundary.

1.3 More complete regularity

The results of this work directly lead to the following regularity result, given by Theorem 3.18 of [24]:

Theorem 3.18 of [24]: *Suppose T is an n -dimensional absolutely area-minimizing integer rectifiable current in an open subset of \mathbb{R}^{n+1} containing the origin, and that near the origin ∂T consists of a sum of $(n - 1)$ -dimensional $C^{1,\alpha}$ orientable submanifolds for some $\alpha \in (0, 1]$ through the origin, each possibly with multiplicity, which pairwise meet only tangentially (with same orientation). Suppose as well that T has a tangent cone at the origin*

$$\mathbb{C} = M\mathbb{E}^n \llcorner \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_n > 0\} + m\mathbb{E}^n \llcorner \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_n < 0\}$$

where M, m are positive integers with $m \leq M - 1$. Then the support of T near the origin is the graph of a smooth solution to the minimal surface equation $u : \mathbb{R}^n \rightarrow \mathbb{R}$, and the orientation vector of T near the origin corresponds to the upward pointing unit normal of the graph of u .

We make clear that in Theorem 3.18 of [24], we not only assume that ∂T consists of orientable submanifolds (with multiplicity) intersecting tangentially (with same orientation) at the origin, in fact we assume that anywhere a pair of these submanifolds intersect they do so tangentially. In [24] we say such T has $C^{1,\alpha}$ *tangentially immersed boundary*, and there the author studies such T more completely.

As proving Theorem 3.18 of [24] takes some effort in and of itself, we focus here on proving Theorem 2.

1.4 Counterexamples

The examples of stable branched minimal immersions given by [27] and [20] show the absolutely area-minimizing hypothesis cannot be relaxed to stability. Indeed, Theorem 1 of [27] holds that if u_0 is a solution to the two-valued minimal surface equation (see the operator \mathcal{M}_0 at the start of §3 of [20]) over the punctured unit disk in \mathbb{R}^2 which can be extended continuously across the origin, then

$$G = \{(re^{i\theta}, u_0(r^{1/2}e^{i\theta/2})) : r \in (0, 1), \theta \in \mathbb{R}\}$$

is a stable minimal immersion with $C^{1,\alpha}$ branch point at $(0, u_0(0))$, for some $\alpha \in (0, 1)$. [27] and [20] show a large non-trivial class of such solutions exist. We can thus show there is a solution u_0 to the two-valued minimal surface equation which can be extended continuously across the origin, so that $\{(re^{i\theta}, u(r^{1/2}e^{i\theta/2})) : r \in (0, 1), \theta \in (0, 3\pi)\}$ satisfies the assumptions of Theorem 2 (with $M = 2, m = 1$, and with the absolutely area-minimizing condition replaced by stability) but fails to satisfy the partial regularity conclusions given there.

Neither does Theorem 2 hold in higher co-dimensions. A counterexample is given by considering the region $\{(re^{i\theta}, r^{3/2}e^{3i\theta/2}) : r > 0, \theta \in [0, 3\pi]\}$ of the holomorphic variety $\{(z, w) : z^3 = w^2\} \subset \mathbb{C} \times \mathbb{C} \cong \mathbb{R}^4$, which is still calibrated and hence area-minimizing.

The best general result in all co-dimensions is thus as in [2], boundary regularity in case of currents with $C^{1,\alpha}$ embedded boundary at points of density near $1/2$; see Theorem 0.1 of [8], which concludes this in fact for almost minimizing currents of arbitrary co-dimension, or more generally, [3] which does this for stationary varifolds. Observe again, that the examples from [27] and [20] show the density = $1/2$ assumption cannot be relaxed without the area-minimizing hypothesis.

1.5 Future work

Clearly, we wish to extend Theorem 2 to the case when $m = 0$, that is when T has tangent cone at the origin

$$\mathbb{C} = ME^n \llcorner \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_n > 0\}$$

for M a positive integer. What prevents us from modifying the work of [14] in case $m = 0$ is specifically (A.0.8), which is only generally true if $m \geq 1$.

Naturally, we also wish to give a similar result to Theorem 2 in all cases, without assuming T has a tangent cone at the origin of a certain form. At least, we make the following:

Conjecture: *Suppose T is an n -dimensional absolutely area-minimizing integer rectifiable current in an open subset of \mathbb{R}^{n+1} containing the origin, and that near the origin ∂T consists of a sum of $(n - 1)$ -dimensional $C^{1,\alpha}$ orientable submanifolds for some $\alpha \in (0, 1]$, each possibly with multiplicity, meeting tangentially (with same orientation) at the origin. Then T has unique tangent cone at the origin.*

To prove this, it may be necessary to apply the techniques of [31], which gives a general regularity theory for co-dimension one stable minimal hypersurfaces. We suspect this as [31] proceeds by considering the geometric structure of a co-dimension one stable minimal hypersurface near points where such a hypersurface has a tangent cone consisting of a sum of half-hyperplanes meeting along a common co-dimension two subspace (the “spine” of the tangent cone). Moreover, [31] carries through this examination in part by generalizing the techniques of [14], albeit to an exceedingly sophisticated degree.

1.6 Summary

We now discuss the organization of this work.

Our aim is to extend Corollary 9.3 of [14] to the conditions set forth by Theorem 2. This involves making small but ubiquitous changes to the proofs found in [14], up to Theorem 11.1 found therein. This task is undertaken in sections 3-13. In section 3, in particular in sections 3.1, 3.2, we introduce our notation, which is different and more modern than the notation used in [14]. To modify [14], we must rely on the calculations established in the Appendix, which contain the deeper differences between the present setting and the proof of [14].

Each section of [14] is devoted to a large theoretical step, further divided into subsections, given either by closely related computations, lemma, or theorem. We follow the same general structure as well. We include minor,

although clarifying, corrections to [14]. We will also include expository comments, some taken from [7].

Before all this, we state in section 2 the main result Theorem 2, giving exactly the assumptions necessary. Corollary 2 concludes uniqueness of tangent cones for T satisfying the conditions of Theorem 2. We also remark in Theorem 2, using [4], why currents T as in Theorem 2 must have tangent cones at such tangential boundary points.

2 Main results

For the definition of absolutely area-minimizing, consult 5.1.6 of [10]. We denote $\mathbb{E}^{n-1}, \mathbb{E}^n$ to be the currents associated respectively to $\mathbb{R}^{n-1} \times \{0\}, \mathbb{R}^n \times \{0\}$ in \mathbb{R}^{n+1} each with usual orientation, as in 4.1.7 [10]. Given $r > 0$, we denote the homothety $\eta_{0,\rho}(x) = x/\rho$, and for a current T we let $\eta_{0,\rho\#}T$ be the push-forward of T by $\eta_{0,\rho}$. Let also $\text{Clos } A$ denote the closure of $A \subseteq \mathbb{R}^{n+1}$. We now state our main result.

Theorem. *Suppose $\alpha \in (0, 1]$ and T is an n -dimensional absolutely area-minimizing locally rectifiable integer multiplicity current in $\{x \in \mathbb{R}^{n+1} : |x| < 3\}$. We also suppose T satisfies the hypothesis:*

$$\begin{aligned}
 & \partial T \llcorner \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : |(x_1, \dots, x_n)| < 2, |x_{n+1}| < 2\} \\
 (*) \quad & = (-1)^n \sum_{\ell=1}^N m_\ell \Phi_{T,\ell\#}(\mathbb{E}^{n-1} \llcorner \{z \in \mathbb{R}^{n-1} : |z| < 2\}),
 \end{aligned}$$

where m_ℓ are positive integers, and for each $\ell \in \{1, \dots, N\}$ and $z \in \mathbb{R}^{n-1}$ with $|z| < 2$

$$\Phi_{T,\ell}(z) = (z, \varphi_{T,\ell}(z), \psi_{T,\ell}(z))$$

where $\varphi_{T,\ell}, \psi_{T,\ell} \in C^{1,\alpha}(\{z \in \mathbb{R}^{n-1} : |z| < 2\})$ with $\varphi_{T,\ell}(0) = 0 = \psi_{T,\ell}(0)$, $D\varphi_{T,\ell}(0) = 0 = D\psi_{T,\ell}(0)$.

(**) T has a tangent cone

$$M\mathbb{E}^n \llcorner \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_n > 0\} + m\mathbb{E}^n \llcorner \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_n < 0\}$$

at the origin, where $M \in \{2, 3, \dots\}$ and $m \in \{1, \dots, M-1\}$.

Then there is a $\delta = \delta(n, M, m, \alpha) \in (0, 1)$ sufficiently small, so that letting

$$\begin{aligned}\tilde{V} &= \{y = (y_1, \dots, y_n) \in \mathbb{R}^n : y_n > |y|^{1+\frac{\alpha}{4n+6}}, |y| < \delta\} \\ \tilde{W} &= \{y = (y_1, \dots, y_n) \in \mathbb{R}^n : y_n < -|y|^{1+\frac{\alpha}{4n+6}}, |y| < \delta\},\end{aligned}$$

then for $\rho > 0$ sufficiently small depending on T we have

$$\begin{aligned}\mathbf{p}^{-1}(\tilde{V}) \cap \text{spt } \boldsymbol{\eta}_{0,\rho\sharp} T &= \text{graph}_{\tilde{V}} \tilde{v} \\ \mathbf{p}^{-1}(\tilde{W}) \cap \text{spt } \boldsymbol{\eta}_{0,\rho\sharp} T &= \text{graph}_{\tilde{W}} \tilde{w}\end{aligned}$$

for some $\tilde{v} \in C^{1, \frac{\alpha}{4n+6}}(\text{Clos } \tilde{V})$, $\tilde{w} \in C^{1, \frac{\alpha}{4n+6}}(\text{Clos } \tilde{W})$ such that $\tilde{v}|_{\tilde{V}}, \tilde{w}|_{\tilde{W}}$ satisfy the minimal surface equation with $D\tilde{v}(0) = 0 = D\tilde{w}(0)$. Furthermore, we have

$$\sup_{y \in \tilde{V}} \frac{|D^2 \tilde{v}(y)|}{|y|^{\frac{\alpha}{4n+6}-1}} + \sup_{y \in \tilde{W}} \frac{|D^2 \tilde{w}(y)|}{|y|^{\frac{\alpha}{4n+6}-1}} \leq c$$

for some $c = c(n, M, m) \in (0, \infty)$.

Note that $M - m = \sum_{\ell=1}^N m_\ell$. As noted in the introduction, the case $m = M - 1$ is just Corollary 9.3 of [14], together with Lemma 10.1 of [14]. Also, if $N = 1$, that is when ∂T is an $(n - 1)$ -dimensional $C^{1,\alpha}$ submanifold with multiplicity, then Theorem 2 follows in this case by the higher multiplicity boundary regularity given by [30]. Nonetheless, the proof we give below will cover all cases. The following corollary immediately follows.

Corollary. *If T is as in Theorem 2, then T has unique tangent cone*

$$M\mathbb{E}^n \llcorner \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_n > 0\} + m\mathbb{E}^n \llcorner \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_n < 0\}$$

at the origin.

Before proceeding, we prove a lemma showing the existence of tangent cones to start.

Lemma. *Suppose $\alpha \in (0, 1]$ and T is an n -dimensional absolutely area-minimizing locally rectifiable integer multiplicity current in $\{x \in \mathbb{R}^{n+1} : |x| < 3\}$ satisfying hypothesis $(*)$. Then T has an oriented tangent cone at the origin, and every oriented tangent cone of T at the origin is absolutely area minimizing with density at the origin equal to the density of T at the origin.*

Proof. By Theorems 3.6, 3.3 of [4] we only need to check the finiteness of

$$\nu_1^{\partial T}(0) = \int_{\{x \in \mathbb{R}^{n+1} : |x| < 1\}} \frac{|\partial \vec{T} \wedge x|}{|x|^n} d\mu_{\partial T}(x).$$

This follows by hypothesis (*). \square

Observe of course that T may satisfy hypothesis (*) but not (**), if for example T is a union of half-planes in space, appropriately oriented.

3 Notation and preliminaries

While it is tempting to use the same notation as [14], we take this opportunity to offer cleaner, modern notation. This will not inconvenience the reader, as this article can be read independently of [14].

For $\mathbb{N} = \{1, 2, \dots\}$, in what follows we fix numbers

$$n, M \in \mathbb{N} \text{ with } n \geq 2, \ m \in \{0\} \cup \mathbb{N} \text{ with } m \leq M - 1, \ \alpha \in (0, 1].$$

In this section we will also use $\tilde{n} \in \{1, \dots, n + 1\}$. Observe that while we will only show Theorem 2 holds with $m \in \mathbb{N}$, many of the calculations here hold with $m = 0$. In particular, we only use $m \geq 1$ in Lemma 8.4, Theorem 9.3, section 10, Theorem 11.1, section 13, and in (A.0.8).

We will introduce and use constants c_1, \dots, c_{49} , a few of which are exactly as in [14]. We must take care to show that c_1, \dots, c_{46} depend only on n, M, m in order to apply the iterative arguments used to prove the crucial Lemma 11.2. The constants c_{47}, c_{48}, c_{49} will only be used in proving the general Hopf-type boundary point Lemma 12.1; as such, c_{47}, c_{48}, c_{49} are exactly as in [14].

3.1 Notation associated to Euclidean space.

This section combines sections 1.1, 1.2, 1.3 of [14]. We will mostly work in the Euclidean spaces $\mathbb{R}^{n-1}, \mathbb{R}^n, \mathbb{R}^{n+1}$.

- We shall typically write points in the three Euclidean spaces

$$\begin{aligned} x &= (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}, \\ y &= (y_1, \dots, y_n) \in \mathbb{R}^n, \\ z &= (z_1, \dots, z_{n-1}) \in \mathbb{R}^{n-1}. \end{aligned}$$

We shall as well use $\tilde{x} \in \mathbb{R}^n, \tilde{y} \in \mathbb{R}^n, \tilde{z} \in \mathbb{R}^{n-1}$. We will also identify \mathbb{R}^n with $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ by identifying $y \in \mathbb{R}^n$ with $(y, 0) \in \mathbb{R}^{n+1}$; we do the same for \mathbb{R}^{n-1} in both \mathbb{R}^n and \mathbb{R}^{n+1} .

- If $A \subseteq \mathbb{R}^{n+1}$, then we shall denote the closure of A by $\text{Clos } A$, and we denote the boundary of A by $\partial A = \text{Clos } A \setminus A$.
- Define the following functions. Fix $\rho \in (0, \infty)$ and $\theta \in \mathbb{R}$.

$$\begin{aligned} \boldsymbol{\eta}_\rho : \mathbb{R}^{n+1} &\rightarrow \mathbb{R}^{n+1}, \quad \boldsymbol{\eta}_\rho(x) = x/\rho. \\ \mathbf{rot}_\theta : \mathbb{R}^{n+1} &\rightarrow \mathbb{R}^{n+1}, \\ \mathbf{rot}_\theta(x) &= (x_1, \dots, x_{n-1}, x_n \cos \theta - x_{n+1} \sin \theta, x_n \sin \theta + x_{n+1} \cos \theta). \\ \mathbf{p} : \mathbb{R}^{n+1} &\rightarrow \mathbb{R}^n, \quad \mathbf{p}(x) = (x_1, \dots, x_n). \\ \mathbf{q}_0 : \mathbb{R}^{n+1} &\rightarrow \mathbb{R}, \quad \mathbf{q}_0(x) = x_n. \\ \mathbf{q}_1 : \mathbb{R}^{n+1} &\rightarrow \mathbb{R}, \quad \mathbf{q}_1(x) = x_{n+1}. \end{aligned}$$

Note that the notation for $\boldsymbol{\eta}_\rho$ is derived from the usual notation for the more general homothety $\boldsymbol{\eta}_{\tilde{x}, \rho}(x) = \frac{x - \tilde{x}}{\rho}$. Since we will only once need the homothety $\boldsymbol{\eta}_{-x, 1}$ (translation by x , in the proof of Theorem 7.4), then we use $\boldsymbol{\eta}_\rho$ for rescaling.

- We denote the following subsets of \mathbb{R}^{n+1} . For $x \in \mathbb{R}^{n+1}$ and $\rho \in (0, \infty)$, we shall denote the open and closed ball by

$$\begin{aligned} B_\rho(x) &= \{\tilde{x} \in \mathbb{R}^{n+1} : |\tilde{x} - x| < \rho\} \\ \bar{B}_\rho(x) &= \{\tilde{x} \in \mathbb{R}^{n+1} : |\tilde{x} - x| \leq \rho\}. \end{aligned}$$

We will mainly consider balls centered at the origin, and so we denote

$$B_\rho = B_\rho(0) \text{ and } \bar{B}_\rho = \bar{B}_\rho(0).$$

Of great importance will be vertical closed cylinders centered at the origin, so we denote

$$C_\rho = \{\tilde{x} \in \mathbb{R}^{n+1} : |\mathbf{p}(\tilde{x})| \leq \rho\}.$$

- Given $z \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}^n$, we denote the open and closed balls in $\mathbb{R}^{n-1}, \mathbb{R}^n$ with $\rho \in (0, \infty)$

$$\begin{aligned} B_\rho^{n-1}(z) &= B_\rho(z) \cap \mathbb{R}^{n-1}, \quad \bar{B}_\rho^{n-1}(z) = \bar{B}_\rho(z) \cap \mathbb{R}^{n-1} \\ B_\rho^n(y) &= B_\rho(y) \cap \mathbb{R}^n, \quad \bar{B}_\rho^n(y) = \bar{B}_\rho(y) \cap \mathbb{R}^n. \end{aligned}$$

We will specifically use balls centered at the origin, denoted by

$$\begin{aligned} B_\rho^{n-1} &= B_\rho \cap \mathbb{R}^{n-1}, & \bar{B}_\rho^{n-1} &= \bar{B}_\rho \cap \mathbb{R}^{n-1} \\ B_\rho^n &= B_\rho \cap \mathbb{R}^n, & \bar{B}_\rho^n &= \bar{B}_\rho \cap \mathbb{R}^n \end{aligned}$$

The following sets will also be important, so we denote them as in [14]:

$$\mathbf{L} = B_1^{n-1}, \mathbf{V} = \{y \in B_1^n : y_n > 0\}, \mathbf{W} = \{y \in B_1^n : y_n < 0\}$$

$$\mathbf{V}_\sigma = \{y \in \mathbf{V} : \text{dist}(y, \partial \mathbf{V}) > \sigma\}, \mathbf{W}_\sigma = \{y \in \mathbf{W} : \text{dist}(y, \partial \mathbf{W}) > \sigma\}$$

for $\sigma \in (0, 1)$.

- Let $e_1, \dots, e_{n+1} \in \mathbb{R}^{n+1}$ be the standard basis vectors; we will also denote the standard basis $e_1, \dots, e_{n-1} \in \mathbb{R}^{n-1}$ and $e_1, \dots, e_n \in \mathbb{R}^n$. We let $*$: $\bigwedge_n \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ be the Hopf map

$$* \left(\sum_{i=1}^{n+1} x_i (-1)^{i-1} e_1 \wedge \dots \wedge e_{i-1} \wedge e_{i+1} \wedge \dots \wedge e_{n+1} \right) = \sum_{k=1}^{n+1} x_k e_k.$$

Note that $*(e_1 \wedge \dots \wedge e_n) = (-1)^n e_{n+1}$.

- Let $\mathcal{H}^{\tilde{n}}$ be \tilde{n} -dimensional Hausdorff measure in \mathbb{R}^{n+1} . Denote $\varpi_{\tilde{n}} = \mathcal{H}^{\tilde{n}}(B_1 \cap \mathbb{R}^{\tilde{n}})$.
- D will denote differentiation over Euclidean space, with dimension clear by context, and D_k partial differentiation with respect to the k^{th} variable.

$\text{Lip}(\phi)$ will denote the Lipschitz constant for a Lipschitz function $\phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$.

We will use various spaces of Hölder continuously differentiable functions. For example, $C_c^{k,a}(A, \tilde{A})$ will denote the k times Hölder continuously differentiable functions of order a with compact support in A and values in \tilde{A} , where $k \in \mathbb{N}$ and $a \in [0, 1]$.

3.2 Notation associated to $T \in \mathcal{R}_n(\mathbb{R}^{n+1})$.

This is section 1.4 of [14]. For a thorough introduction to currents, see [10],[26].

- Recall that $\mathcal{D}^{\tilde{n}}(\mathbb{R}^{n+1})$ denotes the smooth compactly supported \tilde{n} -forms. The \tilde{n} -dimensional currents over \mathbb{R}^{n+1} are then the set of linear functionals over $\mathcal{D}^{\tilde{n}}(\mathbb{R}^{n+1})$.

Given an \tilde{n} -dimensional current T , as usual ∂T will denote the associated boundary of T . So, ∂T is the $(\tilde{n} - 1)$ -dimensional current defined by $\partial T(\omega) = T(d\omega)$ for $\omega \in \mathcal{D}^{\tilde{n}-1}(\mathbb{R}^{n+1})$.

- For T a current over \mathbb{R}^{n+1} and $\phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$, we denote $\phi_{\#}T$ the push-forward current of T by ϕ ; we shall frequently make use of $\eta_{\rho_{\#}}T$.
- Denote by $\mathbb{E}^{\tilde{n}}$ the \tilde{n} -dimensional current in \mathbb{R}^{n+1} given by $\mathbb{E}^{\tilde{n}}(\omega) = \int_{\mathbb{R}^{\tilde{n}}} \langle \omega, e_1 \wedge \dots \wedge e_{\tilde{n}} \rangle d\mathcal{H}^{\tilde{n}}$ for $\omega \in \mathcal{D}^{\tilde{n}}(\mathbb{R}^{n+1})$.
- For T an \tilde{n} -dimensional current in \mathbb{R}^{n+1} , we let μ_T denote the associated mass measure of T ; This is given by $\mu_T(U) = \sup_{\omega \in \mathcal{D}^{\tilde{n}}(U), |\omega| \leq 1} T(\omega)$ when U is an open subset of \mathbb{R}^{n+1} . We define the support of T by $\text{spt } T = \text{spt } \mu_T$. Let \vec{T} be the $\bigwedge_{\tilde{n}}(\mathbb{R}^{n+1})$ -valued orientation of T . Thus, $T(\omega) = \int \langle \omega, \vec{T} \rangle d\mu_T$ for $\omega \in \mathcal{D}^{\tilde{n}}(\mathbb{R}^{n+1})$.

We denote the mass of T by $\mathbf{M}(T) = \mu_T(\mathbb{R}^{n+1})$.

For A a μ_T -measurable set, we let $T \llcorner A$ denote the restriction current $(T \llcorner A)(\omega) = \int_A \langle \omega, \vec{T} \rangle d\mu_T$ for $\omega \in \mathcal{D}^{\tilde{n}}(\mathbb{R}^{n+1})$. In some instances we may use $\mathbf{M}(T \llcorner A)$ instead of $\mu_T(A)$, in order to avoid cluttered notation.

We denote the density of T at $x \in \mathbb{R}^{n+1}$ by $\Theta_T(x) = \lim_{\rho \searrow 0} \frac{\mu_T(B_\rho(x))}{\varpi_{\tilde{n}} \rho^{\tilde{n}}}$, whenever this limit exists.

- We shall let $\mathcal{R}_{\tilde{n}}(\mathbb{R}^{n+1})$ denote the \tilde{n} -dimensional (integer) rectifiable currents over \mathbb{R}^{n+1} .

For $T \in \mathcal{R}_{\tilde{n}}(\mathbb{R}^{n+1})$, denote the approximate (\tilde{n} -dimensional) tangent space of T at x by $T_x T$, for the μ_T -almost-every $x \in \mathbb{R}^{n+1}$ such that this space exists.

- For $T \in \mathcal{R}_n(\mathbb{R}^{n+1})$, we denote the associated generalized unit normal vector field \mathcal{V}^T defined by $\mathcal{V}^T(x) = *\vec{T}(x)$ for μ_T -almost-every x ; note that $\mathcal{V}^T(x) \perp T_x T$. We will also write the components $\mathcal{V}^T = (\mathcal{V}_1^T, \dots, \mathcal{V}_{n+1}^T)$.
- For $f \in C^1(\mathbb{R}^{n+1})$ let $\nabla^T f = Df - (Df \cdot \mathcal{V}^T)\mathcal{V}^T$ denote tangential differentiation over T . For $X \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ let $\operatorname{div}_T X = \operatorname{div}_{\mathbb{R}^{n+1}} X - \mathcal{V}^T \cdot (DX\mathcal{V}^T)$ denote the divergence over T .
- We say $T \in \mathcal{R}_n(\mathbb{R}^{n+1})$ is (absolutely) area minimizing if for any $R \in \mathcal{R}_n(\mathbb{R}^{n+1})$ with $\partial R = \partial T$ we have $\mathbf{M}(T) \leq \mathbf{M}(R)$.

We now define the cylindrical and spherical excess of a current.

Definition. Suppose $T \in \mathcal{R}_n(\mathbb{R}^{n+1})$. We define the cylindrical excess of T at radius $r \in (0, \infty)$ to be

$$\mathbf{E}_C(T, r) = \frac{\mu_T(C_r) - \mu_{\mathbf{p}_\# T}(C_r)}{r^n}.$$

If $\Theta_T(0)$ exists, we define the spherical excess of T at radius $r \in (0, \infty)$ by

$$\mathbf{E}_S(T, r) = \frac{\mu_T(\bar{B}_r)}{r^n} - \varpi_n \Theta_T(0).$$

From the definition of cylindrical excess, as well as Remark 27.2(3) of [26], we get the following monotonicity for $0 < r < s < \infty$

$$\begin{aligned} (3.2.1) \quad r^n \mathbf{E}_C(T, r) &= \int_{C_r} 1 - |\mathcal{V}_{n+1}^T| \, d\mu_T \\ &\leq \int_{C_s} 1 - |\mathcal{V}_{n+1}^T| \, d\mu_T = s^n \mathbf{E}_C(T, s). \end{aligned}$$

This is 1.4(1) of [14], with a slight correction.

3.3 The family \mathcal{I} .

This is sections 1.5, 1.6 of [14] with substantial changes. Most notably, we replace section 1.5 of [14] with assumption (3.3.4).

Definition. Let $M \in \mathbb{N}$, $m \in \{0, \dots, M-1\}$, and $\alpha \in (0, 1]$. Define $\mathcal{T} = \mathcal{T}(M, m, \alpha)$ to be the collection of (absolutely) area minimizing $T \in \mathcal{R}_n(\mathbb{R}^{n+1})$ such that the following four hold.

First, T satisfies the basic support, mass, and density identities

$$(3.3.1) \quad \begin{aligned} \text{spt } T &\subset \bar{B}_3 \\ \mathbf{M}(T) &\leq 3^n(1 + M\varpi_n) \\ \Theta_T(0) &= \frac{M+m}{2}. \end{aligned}$$

Second, there are $N, m_1, \dots, m_N \in \mathbb{N}$ so that

$$(3.3.2) \quad \begin{aligned} \sum_{\ell=1}^N m_\ell &= M - m \text{ and} \\ \partial T \llcorner \{x \in \mathbb{R}^{n+1} : |(x_1, \dots, x_{n-1})| < 2, |x_n| < 2\} \\ &= (-1)^n \sum_{\ell=1}^N m_\ell \Phi_{T, \ell\#}(\mathbb{E}^{n-1} \llcorner B_2^{n-1}) \end{aligned}$$

where for each $\ell = 1, \dots, N$ we have $\Phi_{T, \ell}(z) = (z, \varphi_{T, \ell}(z), \psi_{T, \ell}(z))$ for functions $\varphi_{T, \ell}, \psi_{T, \ell} \in C^{1, \alpha}(B_2^{n-1})$ satisfying $\varphi_{T, \ell}(0) = 0 = \psi_{T, \ell}(0)$, $D\varphi_{T, \ell}(0) = 0 = D\psi_{T, \ell}(0)$.

Third, and moreover, defining

$$(3.3.3) \quad \kappa_T = \frac{2}{\alpha} \max_{\ell=1, \dots, N} \sup_{z \neq \tilde{z}} \frac{|(D\varphi_{T, \ell}(z), D\psi_{T, \ell}(z)) - (D\varphi_{T, \ell}(\tilde{z}), D\psi_{T, \ell}(\tilde{z}))|}{|z - \tilde{z}|^\alpha}$$

then $\kappa_T \leq 1$.

Fourth and final, if we define $\varphi_T^{max}, \varphi_T^{min} : B_2^{n-1} \rightarrow \mathbb{R}$ by

$$\varphi_T^{max}(z) = \max_{\ell=1, \dots, N} \varphi_{T, \ell}(z), \quad \varphi_T^{min}(z) = \min_{\ell=1, \dots, N} \varphi_{T, \ell}(z),$$

then

$$(3.3.4) \quad \begin{aligned} \mathbf{p}_\#(T \llcorner C_2) \llcorner \{y \in B_2^n : y_n \notin [\varphi_T^{min}(y_1, \dots, y_{n-1}), \varphi_T^{max}(y_1, \dots, y_{n-1})]\} \\ = M \mathbb{E}^n \llcorner \{y \in B_2^n : y_n > \varphi_T^{max}(y_1, \dots, y_{n-1})\} \\ + m \mathbb{E}^n \llcorner \{y \in B_2^n : y_n < \varphi_T^{min}(y_1, \dots, y_{n-1})\}. \end{aligned}$$

Using (A.0.7) and then (A.0.6), we compute for $r \in (0, 2]$

$$\begin{aligned}
(3.3.5) \quad \mathbf{E}_S(T, r) &\leq \frac{\mu_T(C_r)}{r^n} - \left(\frac{M+m}{2} \right) \varpi_n \\
&\leq \mathbf{E}_C(T, r) + \frac{\mu_{\mathbf{P}_T^T(C_r)}^T}{r^n} - \left(\frac{M+m}{2} \right) \varpi_n \\
&\leq \mathbf{E}_C(T, r) + (M-m) \varpi_{n-1} r^\alpha \kappa_T.
\end{aligned}$$

This is a modification of 1.6(1) of [14], a bound on the spherical excess by the cylindrical excess. Later, in Lemma 9.1 and Theorem 9.3 we must essentially bound the cylindrical excess by the spherical excess.

4 First variation and monotonicity

This is section 2 of [14], with mostly only changes in presentation. Monotonicity formulas are computed in this section, via the first variation. We introduce the constants c_1, \dots, c_5 .

Throughout this section (and after) we shall write $\mathcal{T} = \mathcal{T}(M, m, \alpha)$ with $M \in \mathbb{N}$, $m \in \{0, \dots, M-1\}$, and $\alpha \in (0, 1]$. Recall that we need $m \geq 1$ only in Lemma 8.4, Theorem 9.3, section 10, Theorem 11.1, section 13, and in (A.0.8).

Starting with this section, if the reader subtracts two to the section number, then one gets the counterpart section, subsection, lemma, theorem, and equation of [14].

4.1 First variation

The following first variation formula is well-known; we sketch the proof. We thus give a more precise version of the first variation formula given in section 2.1 of [14].

Lemma. *For any $T \in \mathcal{T}$, there is a $\mu_{\partial T}$ -measurable vector field $\nu_T : (B_2^n \times \mathbb{R}) \rightarrow \mathbb{R}^{n+1}$ with $\nu_T(x) \perp T_x \partial T$ and $|\nu_T(x)| \leq 1$ for $\mu_{\partial T}$ -almost-every $x \in B_2^n \times \mathbb{R}$ such that*

$$\int \operatorname{div}_T X \, d\mu_T = \int \nu_T \cdot X \, d\mu_{\partial T}$$

for every $X \in C_c^1(B_2^n \times \mathbb{R})$.

Note that (as opposed to [14]) we have used \mathcal{V}^T to denote the generalized unit normal vector field of T , given by $\mathcal{V}^T = *\vec{T}$. In this lemma and here throughout, ν_T denotes the generalized co-normal of ∂T with respect to T . The proof here is taken from the proof of (2.10) of [9].

Proof. Examining the proof of Lemma 3.1 of [4], we conclude $|\int \operatorname{div}_T X \, d\mu_T| \leq \int |X \wedge \vec{\partial T}| \, d\mu_{\partial T}$; to see this, note that in our setting we may take $\lambda = 0$ in the proof of Lemma 3.1 of [4]. This together with section 39 of [26] implies the lemma. \square

4.2 Monotonicity

We give the monotonicity formulas of section 2.2 of [14]. The conclusions bear only changes in notation.

Lemma. *There are constants c_1, c_2, c_3, c_4 depending on n, M, m so that for any $T \in \mathcal{T}$, $0 < r \leq s < 2$, and $\kappa \in (0, 1)$*

$$(4.2.1) \quad \frac{\mu_T(\bar{B}_r)}{r^n} e^{c_1 \kappa_T r^\alpha} \text{ is nondecreasing in } r,$$

$$(4.2.2) \quad \frac{1}{c_2} \leq \frac{\mu_T(\bar{B}_r)}{r^n} \leq c_2,$$

$$(4.2.3) \quad \int_{\bar{B}_2} \frac{1}{|x|^{(1-\kappa)n}} d\mu_T(x) \leq \frac{c_3}{\kappa},$$

$$(4.2.4) \quad \left| \frac{\mu_T(\bar{B}_s)}{s^n} - \frac{\mu_T(\bar{B}_r)}{r^n} - \int_{\bar{B}_s \setminus \bar{B}_r} \frac{|x \cdot \mathcal{V}^T(x)|^2}{|x|^{n+2}} d\mu_T(x) \right| \leq \frac{c_4 \kappa_T}{2} \left[(s^\alpha - r^\alpha) + \alpha \left(1 - \left(\frac{r}{s} \right)^n \right) r^\alpha \right],$$

$$(4.2.5) \quad \left| \mathbf{E}_S(T, s) - \int_{\bar{B}_s} \frac{|x \cdot \mathcal{V}^T(x)|^2}{|x|^{n+2}} d\mu_T(x) \right| \leq c_4 \kappa_T.$$

These are respectively formulas 2.2(1)-(5) of [14], but where c_1, c_2, c_3, c_4 now depend on n, M, m . Observe that our use of $\varkappa \in (0, 1)$ in (4.2.3) differs from that of $\beta \in (0, n)$ as used in 2.2(3) of [14]; in fact, we won't be needing (4.2.3), but present it for potential future use, and to keep our references to equations as close to [14] as feasible.

Proof. We first prove (4.2.4). Consider the Lipschitz vector field

$$X(x) = \begin{cases} (r^{-n} - s^{-n})x & \text{for } |x| \leq r, \\ (|x|^{-n} - s^{-n})x & \text{for } r < |x| \leq s, \\ 0 & \text{for } s < |x|, \end{cases}$$

and recall that by the previous section we have $\int \operatorname{div}_T X \, d\mu_T = \int \nu_T \cdot X \, d\mu_{\partial T}$. Using (3.3.2) we conclude

$$\begin{aligned} \left| \int \operatorname{div}_T X \, d\mu_T \right| &\leq \int |X \wedge \vec{\partial T}| \, d\mu_{\partial T} \\ &= \sum_{\ell=1}^N m_\ell \int_{\Phi_{T,\ell}(B_2^{n-1})} |X \wedge \vec{\partial T}| \, d\mathcal{H}^{n-1} \end{aligned}$$

Furthermore, for each $\ell = 1, \dots, N$ we have by (3.3.3) that for \mathcal{H}^{n-1} -almost-every $x \in \Phi_{T,\ell}(B_2^{n-1})$ (as in 2.2(6) of [14])

$$(4.2.6) \quad \begin{aligned} |x| &\leq (1 + \kappa_T)|(x_1, \dots, x_{n-1})| \leq 2|(x_1, \dots, x_{n-1})|, \\ |x \wedge \vec{\partial T}(x)| &\leq c_5 \kappa \alpha |(x_1, \dots, x_{n-1})|^{1+\alpha}, \end{aligned}$$

where c_5 depends in fact only on n . Thus, (4.2.4) follows by integrating with respect to (x_1, \dots, x_{n-1}) , noting that

$$\begin{aligned} \int_r^s (t^{-n} - s^{-n}) t^{n-1+\alpha} \, dt &\leq \alpha^{-1} (s^\alpha - r^\alpha) \\ (r^{-n} - s^{-n}) \int_0^r t^{n-1+\alpha} \, dt &\leq (1 - (r/s)^n) r^\alpha, \end{aligned}$$

and recalling by (3.3.2) that $\sum_{\ell=1}^N m_\ell = M - m$; from this we get c_4 depending on n, M, m .

Second, define $L : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ (as in [14]) by

$$L(x) = (x_1, \dots, x_{n-1}, |(x_n, x_{n+1})|) \text{ for } x \in \mathbb{R}^{n+1}.$$

Note that $\text{Lip}(L) = 1$ and that $L_{\#}(T \llcorner B_2) \neq 0$ because

$$(\partial L_{\#}(T \llcorner B_2)) \llcorner B_2^n = L_{\#}(\partial T \llcorner B_2) \neq 0.$$

From (4.2.6) and 4.1.31 of [10] we infer that for $r \in (0, 2)$

$$\begin{aligned} r^{-n} \mu_T(\bar{B}_r) &\geq r^{-n} \mu_{L_{\#}T}(\bar{B}_r) \\ &\geq r^{-n} \mathbf{M}(\mathbb{E}^n \llcorner \{y \in \bar{B}_r^n : 2|(y_1, \dots, y_{n-1})| \leq |y|, y_n > 0\}) \\ &= \mathcal{H}^n(\{y \in \bar{B}_r^n : 2|(y_1, \dots, y_{n-1})| \leq |y|, y_n > 0\}). \end{aligned}$$

This along with (4.2.4) implies the first inequality in (4.2.2). Letting $\phi(r) = r^{-n} \mu_T(\bar{B}_r)$, we deduce from this inequality and (4.2.4) that

$$(n+2)c_2c_4\kappa_T\alpha r^{\alpha-1}\phi(r) + \liminf_{s \searrow r} \frac{\phi(s) - \phi(r)}{s - r} \geq 0,$$

which implies (4.2.1) with $c_1 = (n+2)c_2c_4$.

By increasing c_2 if necessary, the second inequality in (4.2.2) now follows from (4.2.1) and (3.3.1); we can also show (4.2.3) from (4.2.2). Meanwhile, (4.2.5) is verified by letting $r \searrow 0$ in (4.2.4). \square

4.3 Remark

We remark here as in section 2.3 of [14]. If $T \in \mathcal{T}$ and $\lambda \in (0, 1/3]$, then (4.2.4) applied with $r = 3\lambda$ and $s = 1$, (3.3.1), and (3.3.5) imply

$$\begin{aligned} \mathbf{M}((\eta_{\lambda\#}T) \llcorner \bar{B}_3) &= 3^n((3\lambda)^{-n} \mu_T(\bar{B}_{3\lambda})) \\ &\leq 3^n(\mu_T(\bar{B}_1) + c_4\kappa_T) \\ &< 3^n \left(\mathbf{E}_C(T, 1) + \left(\frac{M+m}{2} \right) \varpi_n \right) \\ &\quad + 3^n((M-m)\varpi_{n-1} + c_4)\kappa_T. \end{aligned}$$

We can hence choose c_4 depending on n, M, m so that

$\mathbf{E}_C(T, 1) + \kappa_T \leq (1 + c_4)^{-1}$ implies $\mathbf{M}((\eta_{\lambda\#}T) \llcorner \bar{B}_3) < 3^n(1 + M\varpi_n)$. Thus

$$(4.3.1) \quad (\eta_{\lambda\#}T) \llcorner B_3 \in \mathcal{T} \text{ if } \mathbf{E}_C(T, 1) + \kappa_T \leq (1 + c_4)^{-1} \text{ and } \lambda \in (0, 1/3],$$

moreover with

$$(4.3.2) \quad \kappa_{(\eta_{\lambda\#}T) \llcorner B_3} \leq \lambda^\alpha \kappa_T.$$

These are 2.3(1)(2) of [14], with only differences in notation. These equations will be used to iteratively apply results for $T \in \mathcal{T}$ to rescalings of T , most notable in the proof of Lemma 9.1.

5 An area comparison lemma

The results of this section shall be used in the next to conclude preliminary bounds on the excess. This section is analogous to section 3 of [14].

However, we must make a serious change to section 3.2 of [14].

The last constant introduced in section 4 was c_5 , analogous to the last constant c_5 introduced in section 2 of [14]. We note that the first constant introduced in section 3 of [14] is c_7 ; in other words, the constant c_6 is mistakenly skipped. To make it easier for the reader to compare our current work to [14], we as well skip c_6 and introduce c_7, c_8, c_9 . Also, while section 3 of [14] also introduced c_{10} and c_{11} , we will not be needing them.

5.1 Remark

A general fact about exterior algebras is stated, as in section 3.1 of [14].

If $F \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ and $T \in \mathcal{T}$, then for μ_T -almost-every $x \in \mathbb{R}^{n+1}$

$$*((\wedge_n DF)(x)\vec{T}(x)) = \mathcal{V}^T(x)\Delta(x)$$

where $\Delta(x)$ is the $(n+1) \times (n+1)$ matrix with $(i, j)^{\text{th}}$ entry $(-1)^{i+j}$ times the determinant of the $n \times n$ matrix obtained by deleting the i^{th} row and j^{th} column from the $(n+1) \times (n+1)$ matrix with i^{th} row $D_i F(x)$.

5.2 Lemma

We give a lemma analogous to that of section 3.2 of [14]. However, the conclusion of our lemma is slightly different. As such, we must take care that any applications of Lemma 7.2 still follow through as for Lemma 3.2 of [14]. Recall that $\mathbf{q}_1 : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is given by $\mathbf{q}_1(x) = x_{n+1}$ for $x \in \mathbb{R}^{n+1}$. This lemma uses the first variation Lemma 4.1 with “vertical” deformation vector fields, to estimate the change in the mass of T when pushforwarded by certain maps $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ fixing cylinders C_ρ .

Lemma. *There are constants c_7, c_8 depending on n, M, m such that if $T \in \mathcal{T}$, $\rho \in (0, \infty)$, $\tau \in (0, 1)$, $A \subseteq C_{1+\tau}$ is a Borel set satisfying $A = \mathbf{p}^{-1}[\mathbf{p}(A)]$, and $\phi \in C^1(\mathbb{R}^n; [0, 1])$ with $\sup_{\mathbf{p}(A)} |D\phi| \leq \rho/\tau$, then*

$$\mathbf{M}(F_\#(T \llcorner A)) - \mathbf{M}(T \llcorner A) \leq c_7 \kappa_T + \frac{c_8(1 + \rho^2)}{\tau^2} \int_{A_\tau} \mathbf{q}_1^2 d\mu_T,$$

where $F \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ is given by

$$F(x) = (\mathbf{p}(x), \phi(\mathbf{p}(x))x_{n+1}),$$

and where $A_\tau = \{x \in \mathbb{R}^{n+1} : \text{dist}(x, A) < \tau\}$.

Proof. Letting Δ be given as in Remark 5.1, we compute

$$|\mathcal{V}^T \Delta|^2 = \sum_{i=1}^n (\mathcal{V}_i^T \phi - \mathbf{q}_1 \mathcal{V}_i^T D_i \phi)^2 + (\mathcal{V}_{n+1}^T)^2$$

μ_T -almost-everywhere. Using 4.1.30 of [10], Remark 5.1, the inequalities

$$\phi^2 \leq 1, \quad |\mathcal{V}_{n+1}^T| \leq 1, \quad \sum_{i=1}^n (\mathcal{V}_i^T)^2 = 1 - (\mathcal{V}_{n+1}^T)^2,$$

Schwartz's inequality, and Cauchy's inequality ($2ab \leq a^2 + b^2$), we estimate

$$\begin{aligned} & \mathbf{M}(F_{\sharp}(T \llcorner A)) - \mathbf{M}(T \llcorner A) \\ & \leq \int_A |(\wedge_n DF)(x) \vec{T}(x)| \, d\mu_T(x) - \mu_T(A) \\ & = \int_A (|\mathcal{V}^T \Delta| - 1) \, d\mu_T = \int_A \frac{|\mathcal{V}^T \Delta|^2 - 1}{|\mathcal{V}^T \Delta| + 1} \, d\mu_T \\ (5.2.1) \quad & \leq \int_A \left| \sum_{i=1}^n (\mathcal{V}_i^T)^2 \phi^2 - (1 - (\mathcal{V}_{n+1}^T)^2) \right| d\mu_T \\ & \quad + \int_A 2 \left| \mathbf{q}_1 \phi \mathcal{V}_{n+1}^T \sum_{i=1}^n \mathcal{V}_i^T D_i \phi \right| + (\mathbf{q}_1 \mathcal{V}_{n+1}^T)^2 |D\phi|^2 \, d\mu_T \\ & \leq \int_A (1 - (\mathcal{V}_{n+1}^T)^2) + 2|\mathbf{q}_1| \sqrt{1 - (\mathcal{V}_{n+1}^T)^2} |D\phi| + \mathbf{q}_1^2 |D\phi|^2 \, d\mu_T \\ & \leq 2 \int_A (1 - (\mathcal{V}_{n+1}^T)^2) \, d\mu_T + 2 \int_A \mathbf{q}_1^2 |D\phi|^2 \, d\mu_T; \end{aligned}$$

this is 3.2(1) of [14]. The second term is $\leq \frac{2\rho^2}{\tau^2} \int_{A_\tau} \mathbf{q}_1^2 \, d\mu_T$ as is needed.

Consider the first variation formula from Lemma 4.1 with the vector field $X(x) = \zeta(x)^2 \mathbf{q}_1(x) e_{n+1}$ where $\zeta \in C^1(\mathbb{R}^{n+1}; [0, 1])$ satisfies $\text{spt } \zeta \subset A_\tau$,

$\zeta|_A = 1$, and $\sup |D\zeta| \leq \frac{c_9}{\tau}$ with $c_9 = c_9(n)$. We can compute

$$\begin{aligned} & \int (1 - (\mathcal{V}_{n+1}^T)^2) \zeta^2 d\mu_T \\ &= \int (-2\zeta \mathbf{q}_1 \nabla^T \zeta) \cdot e_{n+1} d\mu_T + \int \zeta^2 \mathbf{q}_1 (e_{n+1} \cdot \nu_T) d\mu_{\partial T} \\ &\leq \int \frac{1}{2} (1 - (\mathcal{V}_{n+1}^T)^2) + 2|D\zeta|^2 \mathbf{q}_1^2 d\mu_T + \int \zeta^2 |\mathbf{q}_1| d\mu_{\partial T}, \end{aligned}$$

where recall that ν_T is the co-normal of ∂T with respect to T . Thus

$$\int (1 - (\mathcal{V}_{n+1}^T)^2) \zeta^2 d\mu_T \leq \frac{4c_9^2}{\tau^2} \int_{A_\tau} \mathbf{q}_1^2 d\mu_T + 2 \int \zeta^2 |\mathbf{q}_1| d\mu_{\partial T}.$$

We can also compute

$$\begin{aligned} \int \zeta^2 |\mathbf{q}_1| d\mu_{\partial T} &\leq \int_{C_{1+2\tau}} |\mathbf{q}_1| d\mu_{\partial T} \\ &\leq \left(\frac{\alpha}{2}\right) \kappa_T (1+2\tau)^{1+\alpha} \mu_{\partial T}(C_{1+2\tau}) \\ &\leq 2^{-1} 3^{n+1} \left(1 + \frac{3^2}{4} + \frac{3^4}{16}\right)^{\frac{1}{2}} (M-m) \varpi_{n-1} \kappa_T, \end{aligned}$$

using (3.3.2), (3.3.3), and $\alpha, \kappa_T, \tau \in (0, 1]$ as well as

$$\begin{aligned} & \mu_{\partial T}(C_{1+2\tau}) \\ &\leq (M-m) \varpi_{n-1} \left(1 + \frac{\alpha^2 \kappa_T^2}{4} (1+2\tau)^{2\alpha} + \frac{\alpha^4 \kappa_T^4}{16} (1+2\tau)^{4\alpha}\right)^{1/2} \\ &\quad \times (1+2\tau)^{n-1}. \end{aligned}$$

We conclude the lemma with c_7 depending on n, M, m and c_8 actually just depending on c_9 (and hence, only on n). \square

6 Some preliminary bounds on excess

This section compares the cylindrical excess to the height excess, using subharmonicity while referring to either Theorem 7.5(6) of [1] or Theorem 3.4 of [19]. The proofs and results here are the same as in section 4 of [14],

although we make a slight clarification to the proof found in section 4.1 of [14], given here in section 6.1. In this section we introduce c_{12}, \dots, c_{16} ; in the previous section we introduced c_7, c_8, c_9 while skipping c_{10}, c_{11} , which as opposed to [14] we did not need.

6.1 Lemma

Lemma. *There are positive constants $c_{12}, c_{13}, c_{14}, c_{15} \geq 1$ depending only on n, M, m so that for all $\sigma \in (0, 1)$ and $T \in \mathcal{T}$,*

$$(6.1.1) \quad \begin{aligned} c_{12}^{-1} \sigma^2 \mathbf{E}_C(T, 1) - \kappa_T &\leq \int_{C_{1+\sigma}} \mathbf{q}_1^2 d\mu_T \\ &\leq c_{13} \sup_{C_{1+\sigma} \cap \text{spt } T} \mathbf{q}_1^2; \end{aligned}$$

$$(6.1.2) \quad \begin{aligned} c_{14}^{-1} \sigma^n \sup_{C_{1-\sigma} \cap \text{spt } T} \mathbf{q}_1^2 - \kappa_T^2 &\leq \int_{C_{1-\frac{\sigma}{2}}} \mathbf{q}_1^2 d\mu_T \\ &\leq c_{15} \sigma^{-n-1} (\mathbf{E}_C(T, 1) + \kappa_T). \end{aligned}$$

The equations (6.1.1), (6.1.2) are 4.1(1)(2) of [14]. We as well remark that this lemma means that $\mathbf{E}_C(T, 1)$ can be bounded above and below by fixed multiples of $\sup_{C_{1+\sigma} \cap \text{spt } T} \mathbf{q}_1^2$ and $\sup_{C_{1-\sigma} \cap \text{spt } T} \mathbf{q}_1^2$, respectively, up to a term which is a fixed multiple of κ_T . As remarked in [7], we will repeatedly need to estimate the cylindrical excess by the height. The first equation (6.1.1) follows by the area comparison Lemma 5.2 and a comparison argument based on the minimality of T , while the proof of (6.1.2) uses Allard's technique of Moser iteration; see Theorem 7.5(6) of [1].

Proof. The second inequality in (6.1.1) follows immediately from (3.3.1) with $c_{13} = 3^n(1 + M\varpi_n)$. To prove the first inequality in (6.1.1), first observe that $\kappa_T > 3^n(1 + M\varpi_n)\sigma^2$ implies by (3.3.1) that

$$c_{12}^{-1} \sigma^2 \mathbf{E}_C(T, 1) - \kappa_T < 0$$

so long as we choose $c_{12} \geq 1$.

Now assume $\kappa_T \leq 3^n(1 + M\varpi_n)\sigma^2$. With $\tau = \sigma/2$ let ϕ , F , h , and R_T be as in Lemma A.0.9. Using the homotopy formula (see 4.1.8-9 of [10] or 26.22 of [26]) we can thus compute

$$\partial((T \llcorner C_{1+\tau}) - F_{\sharp}(T \llcorner C_{1+\tau}) - R_T) = \partial(T - F_{\sharp}T - R_T) = 0.$$

Since T is area-minimizing, then this implies

$$\mathbf{M}(T \llcorner C_{1+\tau}) \leq \mathbf{M}(F_{\sharp}(T \llcorner C_{1+\tau}) + R_T).$$

Since $F(x) = \mathbf{p}(x)$ for $x \in C_1$, then letting $A = C_{1+\tau} \setminus C_1$ we compute

$$\begin{aligned} \mathbf{E}_C(T, 1) &= \mathbf{M}(T \llcorner C_1) - \mathbf{M}(F_{\sharp}(T \llcorner C_1)) \\ &\leq \mathbf{M}(F_{\sharp}(T \llcorner C_{1+\tau}) + R_T) - \mathbf{M}(T \llcorner A) - \mathbf{M}(F_{\sharp}(T \llcorner C_1)) \\ &\leq \mathbf{M}(F_{\sharp}(T \llcorner A)) - \mathbf{M}(T \llcorner A) + \mathbf{M}(R_T). \end{aligned}$$

Using (A.0.10) (since $\kappa_T \leq 4 \cdot 3^n(1 + M\varpi_n)\tau^2$), Lemma 5.2 (with $\rho = 3$ and $A = C_{1+\tau} \setminus C_1$), and $\tau = \sigma/2$, gives

$$\begin{aligned} \mathbf{E}_C(T, 1) &\leq c_7 \kappa_T + \frac{40c_8}{\sigma^2} \int_{C_{1+\sigma}} \mathbf{q}_1^2 d\mu_T \\ &\quad + \left(\frac{\sqrt{21}}{8} + 2^{\frac{9n-7}{2}} 3^{n^2-\frac{1}{2}} \right) (M - m) \varpi_{n-1} (1 + M\varpi_n)^{n-1} \kappa_T. \end{aligned}$$

We can then choose $c_{12} \geq 1$, depending on n, M, m so that the first inequality of (6.1.1) holds.

We now prove the first inequality in (6.1.2). For this, we will show the function $\max\{\mathbf{q}_1 - \kappa_T, 0\}^2$ is T -subharmonic, in the sense that

$$\int \nabla^T(\max\{\mathbf{q}_1 - \kappa_T, 0\}^2) \cdot \nabla^T \zeta d\mu_T \leq 0$$

for all $\zeta \in C_c^1(\mathbb{R}^{n+1} \setminus \text{spt } \partial T; [0, \infty))$. In fact, by the first variation formula 4.1, for any such ζ

$$0 = \int \text{div}_T(\zeta e_{n+1}) d\mu_T = \int e_{n+1} \cdot \nabla^T \zeta d\mu_T = \int \nabla^T \mathbf{q}_1 \cdot \nabla^T \zeta d\mu_T.$$

Since $\max\{t - \kappa_T, 0\}^2$ is a nondecreasing convex function and \mathbf{q}_1 is T -harmonic, then by Lemma 7.5(3) of [1] the function $\max\{\mathbf{q}_1 - \kappa_T, 0\}^2$ is T -subharmonic.

Having shown $\max\{\mathbf{q}_1 - \kappa_T, 0\}^2$ is subharmonic, we now use the mean value theorem for subharmonic functions; for this, apply either the argument of Theorem 7.5(6) of [1] (to the varifold associated with T , see

Example 4.8(4) of [1]), or the argument of Theorem 3.4 of [19] (with $\tilde{g}^{ij} = \delta_{ij} - \mathcal{V}_i^T \mathcal{V}_j^T$, $\mu = \mu_T$, and $M = U = B_2^n(0) \times \mathbb{R}$). We deduce the bound

$$\begin{aligned}
(6.1.3) \quad & \sup_{C_{1-\sigma} \cap \text{spt } T} \max\{\mathbf{q}_1 - \kappa_T, 0\}^2 \\
&= \sup_{x \in \mathbf{p}^{-1}(\{0\})} \sup_{B_{1-\sigma}(x) \cap \text{spt } T} \max\{\mathbf{q}_1 - \kappa_T, 0\}^2 \\
&\leq \sup_{x \in \mathbf{p}^{-1}(\{0\})} \frac{c_{16}}{\sigma^n} \int_{B_{1-\frac{\sigma}{2}}(x)} \max\{\mathbf{q}_1 - \kappa_T, 0\}^2 d\mu_T \\
&\leq \frac{c_{16}}{\sigma^n} \int_{C_{1-\frac{\sigma}{2}}} \max\{\mathbf{q}_1 - \kappa_T, 0\}^2 d\mu_T
\end{aligned}$$

where $c_{16} = c_{16}(n) \geq 1$; this is 4.1(3) of [14]. We similarly verify

$$\sup_{C_{1-\sigma} \cap \text{spt } T} \min\{-\kappa_T - \mathbf{q}_1, 0\}^2 \leq \frac{c_{16}}{\sigma^n} \int_{C_{1-\frac{\sigma}{2}}} \min\{-\kappa_T - \mathbf{q}_1, 0\}^2 d\mu_T.$$

Combining the latter two estimates with Cauchy's inequality ($2ab \leq a^2 + b^2$) gives the first inequality in (6.1.2).

To prove the second inequality in (6.1.2), we follow [14] but with small changes in constants. In particular, we now let

$$c_{15} = 16 \cdot 3^{3n+3} (3 + c_4 + (M - m)\varpi_{n-1})(1 + M\varpi_n) \max\{1, \varpi_n^{-1}\} c_{16}.$$

We presently may assume

$$(6.1.4) \quad \mathbf{E}_C(T, 1) + \kappa_T < 3^{n+2} (1 + M\varpi_n) c_{15}^{-1} \sigma^{n+1},$$

analogous to 4.1(4) of [14]. Otherwise using (3.3.1) we get

$$\int_{C_{1-\frac{\sigma}{2}}} \mathbf{q}_1^2 d\mu_T \leq 3^{n+2} (1 + M\varpi_n) \leq c_{15} \sigma^{-n-1} (\mathbf{E}_C(T, 1) + \kappa_T),$$

which is the second inequality in (6.1.2). Assuming (6.1.4), then we use (4.2.5), (3.2.1), (3.3.5) (and assuming $c_4 \geq 3$) to get

$$\begin{aligned}
(6.1.5) \quad & \int_{\bar{B}_1} \mathbf{q}_1^2 d\mu_T = \int_{\bar{B}_1} (x \cdot \mathcal{V}^T(x) + x \cdot (e_{n+1} - \mathcal{V}^T(x)))^2 d\mu_T(x) \\
&\leq 2 \int_{\bar{B}_1} (x \cdot \mathcal{V}^T(x))^2 + |e_{n+1} - \mathcal{V}^T(x)|^2 d\mu_T(x) \\
&\leq 2\mathbf{E}_S(T, 1) + 2c_4\kappa_T + 8\mathbf{E}_C(T, 1) \\
&\leq 2(2 + c_4 + (M - m)\varpi_{n-1})(\mathbf{E}_C(T, 1) + \kappa_T),
\end{aligned}$$

as in 4.1(5) of [14]. Recalling the definition of c_{15} and $\sigma \in (0, 1)$ we conclude

$$\int_{\bar{B}_1} \mathbf{q}_1^2 d\mu_T \leq c_{15} \sigma^{-n-1} (\mathbf{E}_C(T, 1) + \kappa_T).$$

As in [14], we conclude the proof by showing

$$(6.1.6) \quad C_{1-\frac{\sigma}{2}} \cap \text{spt } T \subset \bar{B}_1,$$

as in 4.1(6) of [14]. For this, by the mean value theorem for subharmonic functions as used in (6.1.3), Cauchy's inequality, (6.1.5), and (6.1.4) we have

$$\begin{aligned} \sup_{\bar{B}_{1-\frac{\sigma}{6}} \cap \text{spt } T} \mathbf{q}_1^2 &\leq 2 \cdot 6^n c_{16} \sigma^{-n} \int_{\bar{B}_1} \mathbf{q}_1^2 d\mu_T + 4\kappa_T^2 \\ &\leq 4 \cdot 6^n (3 + c_4 + (M - m) \varpi_{n-1}) c_{16} \sigma^{-n} (\mathbf{E}_C(T, 1) + \kappa_T) \\ &\leq \sigma/12. \end{aligned}$$

This together with (6.1.4) and (3.3.3) implies

$$\partial(T \llcorner B_{1-\frac{\sigma}{6}}) \llcorner C_{1-\frac{\sigma}{3}} = (\partial T) \llcorner C_{1-\frac{\sigma}{3}}.$$

Recalling (6.1.4) (and the definition of c_{15}), we additionally conclude

$$\mathbf{p}_\#((T \llcorner B_{1-\frac{\sigma}{6}}) \llcorner C_{1-\frac{\sigma}{3}}) = \mathbf{p}_\#(T \llcorner C_{1-\frac{\sigma}{3}})$$

using Lemma A.0.11. This together with (3.2.1), the interior monotonicity formula (see 5.4.5(2) of [10] or Theorem 17.6 of [26]), and (6.1.4) imply that for any $x \in (C_{1-\frac{\sigma}{2}} \setminus \bar{B}_1) \cap \text{spt } T$

$$\begin{aligned} \mathbf{E}_C(T, 1) &\geq (1 - \sigma/3) \mathbf{E}_C(T, 1 - \sigma/3) \\ &\geq (2/3)^n \mu_T(B_{\frac{\sigma}{6}}(x)) \\ &\geq (2/3)^n \varpi_n (\sigma/6)^n \geq 3^{-2n} \varpi_n \sigma^{n+1} > \mathbf{E}_C(T, 1). \end{aligned}$$

As this is a contradiction, we must have $(C_{1-\frac{\sigma}{2}} \setminus \bar{B}_1) \cap \text{spt } T = \emptyset$. \square

6.2 Remark

Applying (6.1.2) with $\sigma = \frac{1}{4}$ gives

$$\sup_{C_{\frac{3}{4}} \cap \text{spt } T} |\mathbf{q}_1| \leq \frac{1}{8}$$

whenever $T \in \mathcal{T}$ and $\mathbf{E}_C(T, 1) + \kappa_T \leq 4^{-2n-4} c_{15}^{-1} (1 + c_{14})^{-1}$. Recall the function \mathbf{rot}_θ as given in section 3.1, and observe that the sets

$$\begin{aligned} & \{x \in \mathbb{R}^{n+1} : |(x_1, \dots, x_{n-1})| \leq 1/2, |x_n| < 1/2\} \\ & \mathbf{rot}_\theta(\mathbf{q}_1^{-1}([-1/8, 1/8]) \cap \partial B_{3/4}) \end{aligned}$$

do not intersect whenever $|\theta| \leq 1/8$. We conclude from Remark 4.3

$$\begin{aligned} & (\boldsymbol{\eta}_{\frac{1}{4}\#} \mathbf{rot}_{\theta\#} T) \llcorner B_3 = (\mathbf{rot}_{\theta\#} \boldsymbol{\eta}_{\frac{1}{4}\#} T) \llcorner B_3 \in \mathcal{T} \\ (6.2.1) \quad & \text{whenever } |\theta| \leq 1/8, T \in \mathcal{T}, \text{ and} \\ & \mathbf{E}_C(T, 1) + \kappa_T \leq \min \left\{ \frac{1}{(1 + c_4)}, \frac{1}{4^{2n+4} c_{15} (1 + c_{14})} \right\}; \end{aligned}$$

this is 4.2(1) of [14]. For such θ and T , we may use (4.3.2), (6.1.1) (with $\sigma \nearrow 1$), and (6.1.2) (with $\sigma = \frac{1}{4}$) to estimate

$$(6.2.2) \quad \kappa(\boldsymbol{\eta}_{\frac{1}{4}\#} \mathbf{rot}_{\theta\#} T) \llcorner B_3 = \kappa(\boldsymbol{\eta}_{\frac{1}{4}\#} T) \llcorner B_3 \leq 4^{-\alpha} \kappa_T,$$

$$\begin{aligned} (6.2.3) \quad & \mathbf{E}_C((\boldsymbol{\eta}_{\frac{1}{4}\#} \mathbf{rot}_{\theta\#} T) \llcorner B_3, 1) \leq c_{12} c_{13} \sup_{C_2 \cap \text{spt}(\boldsymbol{\eta}_{\frac{1}{4}\#} \mathbf{rot}_{\theta\#} T)} \mathbf{q}_1^2 + c_{12} 4^{-\alpha} \kappa_T \\ & \leq 16 c_{12} c_{13} \sup_{C_{\frac{1}{2}} \cap \text{spt}(\mathbf{rot}_{\theta\#} T)} \mathbf{q}_1^2 + c_{12} 4^{-\alpha} \kappa_T \\ & \leq 16 c_{12} c_{13} (\theta^2 + \sup_{C_{\frac{3}{4}} \cap \text{spt } T} \mathbf{q}_1^2) + c_{12} 4^{-\alpha} \kappa_T \\ & \leq 4^{2n+4} c_{12} (1 + c_{13}) (1 + c_{14}) (1 + c_{15}) \\ & \quad \times (\theta^2 + \mathbf{E}_C(T, 1) + \kappa_T), \end{aligned}$$

as in 4.2(2)(3) of [14]. Finally, combining (6.2.1), (6.2.2), (6.2.3) and Remark 4.3 we conclude that

$$\begin{aligned} & (\boldsymbol{\eta}_{\lambda\#} \mathbf{rot}_{\theta\#} T) \llcorner B_3 \in \mathcal{T} \text{ with } \kappa(\boldsymbol{\eta}_{\lambda\#} \mathbf{rot}_{\theta\#} T) \llcorner B_3 \leq \lambda^\alpha \kappa_T \\ (6.2.4) \quad & \text{whenever } T \in \mathcal{T}, \lambda \in (0, 1/12), \text{ and} \\ & \theta^2 + \mathbf{E}_C(T, 1) + \kappa_T \leq c_{16}^{-1} \end{aligned}$$

where $c_{16} = c_{16}(n, M, m) = 4^{2n+5}(1 + c_4)(1 + c_{12})(1 + c_{13})(1 + c_{14})(1 + c_{15})$; this is as in 4.2(4) of [14].

As remarked in [7], one typically uses Lemma 6.1 to see that if $T \in \mathcal{T}$ with $\mathbf{E}_C(T, 1) + \kappa_T$ small, then slightly tilting and rescaling T yields another member of \mathcal{T} with small excess of order $\mathbf{E}_C(T, 1)$.

7 Interior nonparametric estimates

Section 5.1 of [14] proves a general decomposition theorem, while sections 5.2, 5.3 of [14] state the well-known gradient estimates for solutions to the minimal surface equation. Section 5.4 of [14], which proves an approximate graphical decomposition for $T \in \mathcal{T}$ with sufficiently small cylindrical excess, passes with no serious changes. Sections 7.1-7.4 are direct counterparts to sections 5.1-5.4 of [14], with only minor mostly notational changes. We introduce in this section c_{17}, \dots, c_{25} .

7.1 Lemma

The following lemma concerns general rectifiable currents, and so passes completely unchanged as in section 5.1 of [14]. We will use this to prove Theorem 7.4, which states that T with $\mathbf{E}_C(T, 1) + \kappa_T$ sufficiently small can be respectively decomposed into a sum of graphs over large regions of \mathbf{V} and \mathbf{W} .

Lemma. *If V is an open subset of \mathbb{R}^n , $S \in \mathcal{R}_n(\mathbb{R}^{n+1})$, $S \llcorner \mathbf{p}^{-1}(V) = S$, $(\partial S) \llcorner \mathbf{p}^{-1}(V) = 0$, $\mathbf{m} \in \mathbb{N}$, $\mathbf{p}_\# S = \mathbf{m}(\mathbb{E}^n \llcorner V)$, and $\mathbf{M}(S) - \mathbf{M}(\mathbf{p}_\# S) < \mathcal{H}^n(V)$, then for each $i \in \{1, \dots, \mathbf{m}\}$ there exists $S_i \in \mathcal{R}_n(\mathbb{R}^{n+1})$ so that*

$$\mathbf{p}^{-1}(V) \cap \text{spt } \partial S_k = \emptyset, \quad \mathbf{p}_\# S_k = \mathbb{E}^n \llcorner V, \quad S = \sum_{i=1}^{\mathbf{m}} S_i, \quad \mu_S = \sum_{i=1}^{\mathbf{m}} \mu_{S_i}.$$

Proof. Choose $s > \sup_{\text{spt } S} \mathbf{q}_1$ and let $h : \mathbb{R} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ be given by

$$h(t, x) = (\mathbf{p}(x), (1 - t)s + tx_{n+1})$$

for $(t, x) \in \mathbb{R} \times \mathbb{R}^{n+1}$. Then by 4.1.9, 4.5.17 of [10] we can choose Lebesgue

measurable sets $M_i \subset \mathbb{R}^{n+1}$, for each integer i , so that $M_i \subset M_{i-1}$ and

$$\begin{aligned}\partial h_{\sharp}((\mathbb{E}^1 \llcorner [0, 1]) \times S) &= \sum_{i=-\infty}^{\infty} \partial(\mathbb{E}^{n+1} \llcorner M_i) \\ \mu_{\partial h_{\sharp}((\mathbb{E}^1 \llcorner [0, 1]) \times S)} &= \sum_{i=-\infty}^{\infty} \mu_{\partial(\mathbb{E}^{n+1} \llcorner M_i)};\end{aligned}$$

see 4.1.8 of [10] or Definition 26.16 of [26]. Defining

$$R_i = \partial(\mathbb{E}^{n+1} \llcorner M_i) \llcorner \mathbf{p}^{-1}(V) \cap \{x \in \mathbb{R}^{n+1} : \mathbf{q}_1(x) < s\},$$

where recall $\mathbf{q}_1(x) = x_{n+1}$ for $x \in \mathbb{R}^{n+1}$, then

$$\begin{aligned}\mathbf{p}_{\sharp} R_i &= -\mathbf{p}_{\sharp}(\partial(\mathbb{E}^{n+1} \llcorner M_i) \llcorner \{x \in \mathbb{R}^{n+1} : \mathbf{q}_1(x) \geq s\}) \in \{0, \pm \mathbb{E}^n \llcorner V\}, \\ S &= \sum_{i=-\infty}^{\infty} R_i, \quad \mu_S = \sum_{i=-\infty}^{\infty} \mu_{R_i}.\end{aligned}$$

Letting $I = \{i : \mathbf{p}_{\sharp} R_i \neq 0\}$, we infer that $\#I = \mathbf{m}$ and hence $\mathbf{p}_{\sharp} R_i = \mathbb{E}^n \llcorner V$ for $i \in I$, because

$$\begin{aligned}\mathbf{m} \mathcal{H}^n(V) &= \mathbf{M}(\mathbf{p}_{\sharp} S) = \mathbf{M}\left(\sum_{i \in I} \mathbf{p}_{\sharp} R_i\right) \leq \sum_{i \in I} \mathbf{M}(\mathbf{p}_{\sharp} R_i) \\ &= (\#I) \mathcal{H}^n(V) \leq \sum_{i \in I} \mathbf{M}(R_i) = \mathbf{M}(S) < (\mathbf{m} + 1) \mathcal{H}^n(V).\end{aligned}$$

Moreover, $M_i \subset M_{i-1}$ for all i implies $I = \{\mathbf{i} + 1, \dots, \mathbf{i} + \mathbf{m}\}$, where $\mathbf{i} = (\inf_{i \in I} i) - 1$. Setting

$$S_1 = \sum_{i=-\infty}^{\mathbf{i}+1} R_i, \quad S_2 = R_{\mathbf{i}+2}, \dots, S_{\mathbf{m}-1} = R_{\mathbf{i}+\mathbf{m}-1}, \quad S_{\mathbf{m}} = \sum_{i=\mathbf{i}+\mathbf{m}}^{\infty} R_i,$$

we conclude the lemma. \square

7.2 Remark

We introduce standard L^2 gradient and DeGiorgi-Nash Hölder continuity estimates for uniformly elliptic partial differential equations. We introduce constants c_{17}, c_{18} as in [14], which in fact depend only on n . These estimates are now well-known, but we give them again for convenience.

Lemma. *There exist constants c_{17}, c_{18} depending only on n such that if $\lambda \in (0, 1)$, $\rho \in (0, \infty)$, $y \in \mathbb{R}^n$, $a_{kl} : B_\rho^n(y) \rightarrow \mathbb{R}$ are Lebesgue measurable functions for each $k, l \in \{1, \dots, n\}$ satisfying $a_{kl} = a_{lk}$,*

$$(7.2.1) \quad \lambda |\xi|^2 \leq \sum_{k,l=1}^n a_{kl} \xi_k \xi_l \leq \lambda^{-1} |\xi|^2 \text{ whenever } \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n,$$

and u is a weak solution of

$$\sum_{k,l=1}^n D_l(a_{kl} D_k u) = 0$$

(with u, Du locally square integrable) over $B_\rho^n(y)$, then

$$(7.2.2) \quad \int_{\bar{B}_{\frac{3\rho}{4}}^n(y)} |Du|^2 d\mathcal{H}^n \leq \frac{c_{17}}{\lambda^4 \rho^2} \int_{B_\rho^n(y)} u^2 d\mathcal{H}^n,$$

$$(7.2.3) \quad \sup_{\bar{B}_{\frac{\rho}{2}}^n(y)} u^2 \leq \frac{c_{17}}{\lambda^{2n} \rho^n} \int_{B_{\frac{3\rho}{4}}^n(y)} u^2 d\mathcal{H}^n, \text{ and}$$

$$(7.2.4) \quad |u(w) - u(\tilde{w})| \leq c_{18} \left(\sup_{B_{\frac{\rho}{2}}^n(y)} |u| \right) \left(\frac{|w - \tilde{w}|}{\rho} \right)^\varkappa \text{ for } w, \tilde{w} \in \bar{B}_{\frac{\rho}{4}}^n(y)$$

where $\varkappa = \varkappa(n, \lambda) \in (0, 1)$.

Proof. See Lemma 1, Theorem 1 of [18]. □

We remark that (7.2.1)-(7.2.4) are 5.2(1)-(4) of [14]. In particular, inequality (7.2.4) is the DeGiorgi-Nash estimate.

7.3 Lemma

This section, analogous to section 5.3 of [14] introduces the well-known gradient estimates for solutions to the minimal surface equation. In [14] these estimates are derived for the sake of completeness using the results of section 5.2 there in; we do the same. The constants $c_{19}, c_{20}, c_{21}, c_{22}, c_{23}$ are introduced in this section, which remain unchanged as in [14], depending only on n . We give the following lemma, which is merely Remark 5.3 of [14].

Lemma. *There exist constants $c_{19}, c_{20} \geq 1$ depending on n so that for any solution v of the minimal surface equation on an open subset Ω of \mathbb{R}^n and any point $y \in \Omega$,*

$$(7.3.1) \quad \sup_{\bar{B}_{\frac{\rho}{2}}^n(y)} |Dv| < c_{19} \left(\sup_{\Omega} |v|/\rho \right) \exp(c_{20} \sup_{\Omega} |v|/\rho)$$

where $\rho = \text{dist}(y, \partial\Omega)$.

We note that (7.3.1) is 5.3(1) of [14]. For a more modern reference, see Theorem 16.5 of [12].

Proof. For $n = 2$ this was proved in Theorem 1 of [16]. In general, for $n \geq 2$, we consider two cases.

First, suppose $\sup_{\Omega} |v|/\rho \geq 1$. Here, (7.3.1) follows precisely from the main estimate of [6] (see also [29],[25]).

Second, suppose $\sup_{\Omega} |v|/\rho < 1$. Here the main estimate of [6] implies that $\sup_{\Omega} |Dv| \leq c_{21}$ for some $c_{21} = c_{21}(n)$. We may then choose an appropriate $\lambda = \lambda(c_{21}) \in (0, 1)$ so that we may first apply (7.2.3) with $u = D_{\tilde{k}}v$, for each $\tilde{k} = 1, \dots, n$, and

$$a_{kl} = \frac{1}{\sqrt{1 + |Dv|^2}} \left(\delta_{kl} - \frac{D_k v D_l v}{1 + |Dv|^2} \right),$$

and second apply (7.2.2) with $u = v$ and $a_{kl} = \frac{\delta_{kl}}{\sqrt{1 + |Dv|^2}}$ to conclude that

$$\begin{aligned} \sup_{\bar{B}_{\frac{\rho}{2}}^n(y)} |Dv|^2 &\leq \frac{c_{22}}{\rho^n} \int_{B_{\frac{3\rho}{4}}(y)} |Dv|^2 d\mathcal{H}^n \\ &\leq \frac{c_{23}}{\rho^{n+2}} \int_{B_{\rho}^n(y)} |v|^2 d\mathcal{H}^n \leq c_{23} \varpi_n \sup_{\Omega} |v|^2 / \rho^2, \end{aligned}$$

where c_{22}, c_{23} depend only on n . □

7.4 Theorem

We give a theorem as in section 5.4 of [14]. Note that we make a small, technical correction in defining \mathbf{V}_T and \mathbf{W}_T below. Also, here we conclude

the existence of functions $v_1^T \leq v_2^T \leq \dots \leq v_M^T$ and $w_1^T \leq \dots \leq w_m^T$. As in [14], we introduce $c_{24}, c_{25} \geq 1$ but now depending on n, M, m . We make some clarifications and simplifications to the proof found in [14].

Theorem. *If $m \geq 1$, then there are constants $c_{24}, c_{25} \geq 1$ depending on n, M, m so that for any $T \in \mathcal{T}$ with*

$$\sigma_T = c_{24}(\mathbf{E}_C(T, 1) + \kappa_T)^{\frac{1}{2n+3}} \leq \frac{1}{4},$$

then with

$$\mathbf{V}_T = B_{\frac{1}{4}}^n \cap \mathbf{V}_{\sigma_T} \text{ and } \mathbf{W}_T = B_{\frac{1}{4}}^n \cap \mathbf{W}_{\sigma_T}$$

(recall $\mathbf{V}_\sigma, \mathbf{W}_\sigma$ defined in section 3.1) we have

$$\begin{aligned} \mathbf{p}^{-1}(\mathbf{V}_T) \cap \text{spt } T &= \bigcup_{i=1}^M \text{graph}_{\mathbf{V}_T} v_i^T \\ \mathbf{p}^{-1}(\mathbf{W}_T) \cap \text{spt } T &= \bigcup_{j=1}^m \text{graph}_{\mathbf{W}_T} w_j^T \end{aligned}$$

for some analytic functions $v_i^T \in C^\infty(\mathbf{V}_T)$ and $w_j^T \in C^\infty(\mathbf{W}_T)$ satisfying the minimal surface equation, and such that

$$v_1^T \leq v_2^T \leq \dots \leq v_M^T \text{ and } w_1^T \leq w_2^T \leq \dots \leq w_m^T.$$

Furthermore, for each $i \in \{1, \dots, M\}$, $j \in \{1, \dots, m\}$, and $l \in \{1, 2, 3\}$

$$(7.4.1) \quad |D^l v_i^T(y)| \leq c_{25} \frac{(\mathbf{E}_C(T, 1) + \kappa_T)^{\frac{1}{2}}}{\text{dist}(y, \partial \mathbf{V})^l} \text{ for } y \in \mathbf{V}_T,$$

$$(7.4.2) \quad |D^l w_j^T(y)| \leq c_{25} \frac{(\mathbf{E}_C(T, 1) + \kappa_T)^{\frac{1}{2}}}{\text{dist}(y, \partial \mathbf{W})^l} \text{ for } y \in \mathbf{W}_T,$$

$$\begin{aligned} (7.4.3) \quad & \int_{\mathbf{V}_T} \left(\frac{\partial}{\partial r} \left(\frac{v_i^T(y)}{|y|} \right) \right)^2 |y|^{2-n} d\mathcal{H}^n(y) \\ & + \int_{\mathbf{W}_T} \left(\frac{\partial}{\partial r} \left(\frac{w_j^T(y)}{|y|} \right) \right)^2 |y|^{2-n} d\mathcal{H}^n(y) \\ & \leq 4(\mathbf{E}_S(T, 1) + c_4 \kappa_T) \leq 4\mathbf{E}_C(T, 1) + 4((M - m)\varpi_{n-1} + c_4)\kappa_T, \end{aligned}$$

where $\frac{\partial}{\partial r}f(y) = \frac{y}{|y|} \cdot Df(y)$ and c_4 is as in (4.2.4).

If $m = 0$, then we still conclude the existence of $v_1^T, \dots, v_M^T \in C^\infty(\mathbf{V}_T)$ satisfying the corresponding properties above.

Note that the equations here are analogous to 5.4(1)(2)(3) of [14].

Proof. Suppose $m \geq 1$, and let $\epsilon \in (0, 1)$ be as in 5.3.14 of [10] with λ, κ, m, n replaced by $1, 1, n, n + 1$. Recall c_{14}, c_{15} from (6.1.2), which depend only on n, M, m . Also recall $c_{19}, c_{20} \geq 1$ from Remark 7.3, which depend only on n . We can thus choose $c_{24} = c_{24}(n, M, m) \geq 1$ so that

$$\left(\frac{\sigma}{c_{24}}\right)^{2n+3} < \min \left\{ \frac{\sigma^{2n+3}}{c_{14}(1 + c_{15})c_{19}^2 \exp(c_{20})}, \mathcal{H}^n(\mathbf{V}_{\frac{\sigma}{3}}), \frac{\epsilon}{2^n} \right\}$$

for any $\sigma \in (0, 1)$. With this choice of c_{24} we now fix a current $T \in \mathcal{T}$ for which $\sigma = \sigma_T = c_{24}(\mathbf{E}_C(T, 1) + \kappa_T)^{\frac{1}{2n+3}} \leq \frac{1}{4}$.

First, we wish to apply Lemma 7.1. Using $c_{24} \geq 1$ and $\sigma \leq \frac{1}{4}$, we find that $\kappa_T < \frac{\sigma}{3}$. We conclude

$$(\text{spt } \partial T) \cap \mathbf{p}^{-1}(\mathbf{V}_{\frac{\sigma}{3}} \cap \mathbf{W}_{\frac{\sigma}{3}}) = \emptyset.$$

Estimating $\mathbf{E}_C(T, 1) + \kappa_T = (\sigma/c_{24})^{2n+3}$ by the second quantity in the minimum above, we apply Lemma 7.1 with V, \mathbf{m}, S replaced by $\mathbf{V}_{\frac{\sigma}{3}}, M, T \llcorner \mathbf{p}^{-1}(\mathbf{V}_{\frac{\sigma}{3}})$ (and respectively, $\mathbf{W}_{\frac{\sigma}{3}}, m, T \llcorner \mathbf{p}^{-1}(\mathbf{W}_{\frac{\sigma}{3}})$) to obtain corresponding S_i for $i = 1, \dots, M$ (respectively S_j for $j = 1, \dots, m$) which are each absolutely area minimizing.

Second, we wish to apply interior regularity for area-minimizing currents. Estimating by the first quantity, we get using (6.1.2) and $\sigma \leq \frac{1}{4}$

$$\sup_{C_{1-\frac{\sigma}{3}} \cap \text{spt } T} \mathbf{q}_1 \leq (3^{n+1} \sigma^{n+2})^{\frac{1}{2}} < 1/2.$$

We now estimate by the third quantity and apply 5.3.15 of [10] with $\lambda, \kappa, m, n, r, s, S$ replaced by

$$1, 1, n, n + 1, \left(\frac{\sigma}{c_{24}}\right)^{2n+3}, \frac{1}{2} \left(\frac{\sigma}{c_{24}}\right)^{2n+3}, \boldsymbol{\eta}_{-x, 1\#} S_i$$

for $x \in \mathbf{p}^{-1}(\mathbf{V}_{\frac{2\sigma}{3}}) \cap \text{spt } S_i$ (and respectively, $\boldsymbol{\eta}_{-x,1\sharp} S_j$ for $x \in \mathbf{p}^{-1}(\mathbf{W}_{\frac{2\sigma}{3}}) \cap \text{spt } S_j$) to conclude that $\mathbf{p}^{-1}(\mathbf{V}_{\frac{2\sigma}{3}}) \cap \text{spt } T$ (respectively, $\mathbf{p}^{-1}(\mathbf{W}_{\frac{2\sigma}{3}}) \cap \text{spt } T$) partitions into graphs of (at most) M (respectively, m) solutions of the minimal surface equation over $\mathbf{V}_{\frac{2\sigma}{3}}$ (respectively, $\mathbf{W}_{\frac{2\sigma}{3}}$); note that estimating by the first quantity and $\sigma \leq \frac{1}{4}$ gives $(\sigma/c_{24})^{2n+3} \leq \frac{\sigma}{3}$, and that $\boldsymbol{\eta}_{-x,1}$ is translation by x .

Third, we show (7.4.1) (respectively, similarly (7.4.2)). Fix $i \in \{1, \dots, M\}$ and abbreviate $v = v_i^T$. Using (6.1.2) (with $\sigma = \frac{1}{2}$ therein) gives

$$\sup_{B_{\frac{1}{2}}^n \cap \mathbf{V}_{\frac{2\sigma}{3}}} |v| \leq 2^{\frac{n+1}{2}} c_{14}^{\frac{1}{2}} (1 + c_{15})^{\frac{1}{2}} (\mathbf{E}_C(T, 1) + \kappa_T)^{\frac{1}{2}}$$

Choose any $y \in \mathbf{V}_T = B_{\frac{1}{2}}^n \cap \mathbf{V}_{\sigma_T}$, then using (7.3.1) with $\Omega = \mathbf{V}_{\frac{2\sigma}{3}}$ and $\rho = \text{dist}(y, \partial(B_{\frac{1}{2}}^n \cap \mathbf{V}_{\frac{2\sigma}{3}})) \geq \max\{\frac{\sigma}{3}, \frac{1}{3} \text{dist}(y, \partial \mathbf{V})\}$, as well as using $\sigma = c_{24}(\mathbf{E}_C(T, 1) + \kappa_T)^{\frac{1}{2n+3}} \leq \frac{1}{4}$ and $c_{24} \geq 1$, we conclude

$$\begin{aligned} |Dv(y)| &< c_{19} \left(\sup_{B_{\frac{1}{2}}^n \cap \mathbf{V}_{\frac{2\sigma}{3}}} |v|/\rho \right) \exp \left(c_{20} \sup_{B_{\frac{1}{2}}^n \cap \mathbf{V}_{\frac{2\sigma}{3}}} |v|/\rho \right) \\ &\leq \left(\frac{3c_{19} \sup_{B_{\frac{1}{2}}^n \cap \mathbf{V}_{\frac{2\sigma}{3}}} |v|}{\text{dist}(y, \partial \mathbf{V})} \right) \exp \left(3c_{20} \sup_{B_{\frac{1}{2}}^n \cap \mathbf{V}_{\frac{2\sigma}{3}}} |v|/\sigma \right) \\ &\leq \left(\frac{3 \cdot 2^{\frac{n+1}{2}} c_{14}^{\frac{1}{2}} (1 + c_{15})^{\frac{1}{2}} c_{19} (\mathbf{E}_C(T, 1) + \kappa_T)^{\frac{1}{2}}}{\text{dist}(y, \partial \mathbf{V})} \right) \\ &\quad \times \exp \left(3 \cdot 2^{\frac{n+1}{2}} c_{14}^{\frac{1}{2}} (1 + c_{15})^{\frac{1}{2}} c_{20} (\mathbf{E}_C(T, 1) + \kappa_T)^{\frac{1}{2}}/\sigma \right) \\ &\leq \left(\frac{3 \cdot 2^{\frac{n+1}{2}} c_{14}^{\frac{1}{2}} (1 + c_{15})^{\frac{1}{2}} c_{19} (\mathbf{E}_C(T, 1) + \kappa_T)^{\frac{1}{2}}}{\text{dist}(y, \partial \mathbf{V})} \right) \\ &\quad \times \exp \left(3 \cdot 2^{\frac{n+1}{2}} c_{14}^{\frac{1}{2}} (1 + c_{15})^{\frac{1}{2}} c_{20} \sigma^{n+\frac{1}{2}} c_{24}^{-n-\frac{3}{2}} \right) \\ &\leq \left(\frac{3 \cdot 2^{\frac{n+1}{2}} c_{14}^{\frac{1}{2}} (1 + c_{15})^{\frac{1}{2}} c_{19} (\mathbf{E}_C(T, 1) + \kappa_T)^{\frac{1}{2}}}{\text{dist}(y, \partial \mathbf{V})} \right) \\ &\quad \times \exp \left(c_{14}^{\frac{1}{2}} (1 + c_{15})^{\frac{1}{2}} c_{20} \right) \end{aligned}$$

Thus, (7.4.1) holds with $k = 1$ if we choose c_{25} depending on $n, c_{14}, c_{15}, c_{19}, c_{20}, c_{24}$ (and hence on n, M, m). Moreover, each of the partial

derivatives $D_l v$ satisfies a linear divergence structure equation of the type treated in Remark 7.2, with constant $\lambda \in (0, 1)$ depending only on n, M, m (see for example the proof of Remark 7.3). The inequality (7.4.1) with $l = 2, 3$ thus follows from (7.2.4) and the interior Schauder theory (see section 6.3 of [12]) for uniformly elliptic equations with Hölder continuous coefficients (respectively, (7.4.2) also similarly holds).

To prove (7.4.3), consider again $v = v_i^T$ with fixed $i \in \{1, \dots, M\}$. For any $y \in \mathbf{V}_T$, we compute using (6.1.2) (as above with $\sigma = \frac{1}{2}$ therein), $\sigma \leq \frac{1}{4}$, $c_{19}, c_{20} \geq 1$, $|y| \in (\sigma, \frac{1}{4})$, (7.3.1) (as in showing (7.4.1)), and estimating by the first quantity to give

$$\begin{aligned} |y|^2 + |v(y)|^2 &\leq |y|^2 + 2^{n+1} c_{14} (1 + c_{15}) (\mathbf{E}_C(T, 1) + \kappa_T) \\ &\leq |y|^2 + 2^{-3n-1} \sigma^2 \leq 2^{\frac{3}{n+2}} |y|^2 < 1, \\ |Dv(y)|^2 &\leq 9 \cdot 2^{n+1} c_{14} (1 + c_{15}) c_{19}^2 (\sigma/c_{24})^{2n+3} \sigma^{-2} \\ &\quad \times \exp \left(3 \cdot 2^{\frac{n+3}{2}} c_{14}^{\frac{1}{2}} (1 + c_{15})^{\frac{1}{2}} c_{20} (\sigma/c_{24})^{\frac{2n+3}{2}} \sigma^{-1} \right) \leq 1. \end{aligned}$$

Since $\mathcal{V}^T(y, v(y)) = * \vec{T}(y, v(y)) = (-1)^n \frac{(-Dv(y), 1)}{\sqrt{1+|Dv(y)|^2}}$ for $y \in \mathbf{V}_T$, we compute

$$\begin{aligned} (7.4.4) \quad & \int_{\mathbf{V}_T} \left(\frac{\partial}{\partial r} \left(\frac{v(y)}{|y|} \right) \right)^2 |y|^{2-n} d\mathcal{H}^n(y) \\ &= \int_{\mathbf{V}_T} \frac{(y \cdot Dv(y) - v(y))^2}{|y|^{n+2}} d\mathcal{H}^n(y) \\ &\leq 4 \int_{\mathbf{V}_T} \frac{|(y, v(y)) \cdot \mathcal{V}^T(y, v(y))|^2}{(|y|^2 + |v(y)|^2)^{\frac{n}{2}+1}} \sqrt{1 + |Dv(y)|^2} d\mathcal{H}^n(y) \\ &\leq 4 \int_{\bar{B}_1 \cap \mathbf{p}^{-1}(\mathbf{V})} \frac{|x \cdot \mathcal{V}^T(x)|^2}{|x|^{n+2}} d\mu_T(x); \end{aligned}$$

this is exactly as in 5.4(4) of [14]. We also verify (as in 5.4(5) of [14])

$$\begin{aligned} (7.4.5) \quad & \int_{\mathbf{W}_T} \left(\frac{\partial}{\partial r} \left(\frac{w_j^T(y)}{|y|} \right) \right)^2 |y|^{2-n} d\mathcal{H}^n(y) \\ &\leq 4 \int_{\bar{B}_1 \cap \mathbf{p}^{-1}(\mathbf{W})} \frac{|x \cdot \mathcal{V}^T(x)|^2}{|x|^{n+2}} d\mu_T(x) \end{aligned}$$

for each $j \in \{1, \dots, m\}$. Since (4.2.5) and (3.3.5) imply that

$$\begin{aligned} \int_{\bar{B}_1} \frac{|x \cdot \mathcal{V}^T(x)|^2}{|x|^{n+2}} d\mu_T(x) &\leq \mathbf{E}_S(T, 1) + c_4 \kappa_T \\ &\leq \mathbf{E}_C(T, 1) + ((M - m)\varpi_{n-1} + c_4)\kappa_T, \end{aligned}$$

then (7.4.3) follows from (7.4.4) and (7.4.5).

The case $m = 0$ follows similarly. \square

8 Blowup sequences and harmonic blowups

This section introduces blowup sequences and harmonic blowups, with the aim to prove the necessary rigidity result Lemma 8.4. More precisely (as remarked in [7]), we study the limit functions $f_i : \mathbf{V} \rightarrow \mathbb{R}$ and $g_j : \mathbf{W} \rightarrow \mathbb{R}$ obtained from the local graph representations $v_i^{(k)} : \mathbf{V}_{\sigma_{T_k}} \rightarrow \mathbb{R}$ and $w_j^{(k)} : \mathbf{W}_{\sigma_{T_k}} \rightarrow \mathbb{R}$ of a sequence of currents $T_k \in \mathcal{T}$, with excess $\mathbf{E}_C(T_k, 1) \rightarrow 0$ and $\mathbf{E}_C(T_k, 1)^{-1} \kappa_{T_k} \rightarrow 0$ as $k \rightarrow \infty$, by rescaling with a factor $\mathbf{E}_C(T_k, 1)^{-1}$ in the vertical direction and passing to a convergent subsequence. Our aim is to prove the rigidity result Lemma 8.4, which allows us to relate the height of T_k , for large $k \in \mathbb{N}$ with the corresponding quantity associated with the set of limit functions.

This section is analogous to section 6 of [14]. Only minor, mostly notational changes must be made. The only serious change is seen in justifying (8.4.13), which is analogous to 6.4(13) of [14]. Whereas 6.4(13) of [14] is proved using section 3.2 of [14], we must use Lemma 5.2 (which differs from Lemma 3.2 of [14]) to show (8.4.13). We introduce specifically in section 8.4 constants c_{26}, \dots, c_{33} depending on n, M, m .

In section 8.4 it will be necessary to assume $m \geq 1$.

8.1 Definition

We give the same definition of a blowup sequence and harmonic blowup as in [14]. In this case, we must take functions $v_i^{(k)}, f_i$ and $w_j^{(k)}, g_j$ respectively with $i \in \{1, \dots, M\}$ and $j \in \{1, \dots, m\}$, but still require 6.1(1)-(4) from [14] to hold.

Recalling Definition 3.3 and Theorem 7.4, with $m \geq 1$ suppose that for each $k \in \mathbb{N}$, $i \in \{1, \dots, M\}$, and $j \in \{1, \dots, m\}$, we have

$$\begin{aligned} T_k &\in \mathcal{T}, \varepsilon_k = \mathbf{E}_C(T_k, 1)^{\frac{1}{2}}, \kappa_k = \kappa_{T_k} \\ v_i^{(k)} : \mathbf{V} &\rightarrow \mathbb{R}, \quad v_i^{(k)}|_{\mathbf{V}_{T_k}} = v_i^{T_k}, \quad v_i^{(k)}|_{\mathbf{V} \setminus \mathbf{V}_{T_k}} = 0 \\ w_j^{(k)} : \mathbf{W} &\rightarrow \mathbb{R}, \quad w_j^{(k)}|_{\mathbf{W}_{T_k}} = w_j^{T_k}, \quad w_j^{(k)}|_{\mathbf{W} \setminus \mathbf{W}_{T_k}} = 0; \end{aligned}$$

in case $m = 0$ we simply define $w_j^{(k)} \equiv 0$ over \mathbf{W} . This leads to the following definition.

Definition. We say that the sequence $\{T_k\}_{k \in \mathbb{N}} \subset \mathcal{T}$ is a blowup sequence with associated harmonic blowups f_i, g_j if as $k \rightarrow \infty$

$$(8.1.1) \quad \varepsilon_k \text{ converges to zero,}$$

$$(8.1.2) \quad \varepsilon_k^{-2} \kappa_k \text{ converges to zero,}$$

$$(8.1.3) \quad \varepsilon_k^{-2} v_i^{(k)} \text{ converges uniformly on compact subsets of } \mathbf{V} \text{ to } f_i,$$

$$(8.1.4) \quad \varepsilon_k^{-2} w_j^{(k)} \text{ converges uniformly on compact subsets of } \mathbf{W} \text{ to } g_j.$$

These requirements are the same as 6.1(1)-(4) of [14]. Note that when $m = 0$ we simply have $g_j \equiv 0$. It readily follows from the estimates of (7.4.1), (7.4.2), or from 5.3.7 of [10], that the functions $f_i : \mathbf{V} \rightarrow \mathbb{R}$, $g_j : \mathbf{W} \rightarrow \mathbb{R}$ are harmonic. Moreover, by (6.1.2),

$$(8.1.5) \quad \sup_{\mathbf{V} \cap B_\rho^n} |f_i|^2 + \sup_{\mathbf{W} \cap B_\rho^n} |g_j|^2 \leq 2c_{14}(1 + c_{15})(1 - \rho)^{-2n-1}$$

for each $\rho \in (0, 1)$; see 6.1(5) of [14]. We will frequently use, by (8.1.5), (7.4.1), (7.4.2), and the Arzela-Ascoli theorem, the following fact:

Lemma. Every sequence $\{T_k\}_{k \in \mathbb{N}} \subset \mathcal{T}$ for which

$$\lim_{k \rightarrow \infty} (\mathbf{E}_C(T_k, 1) + \mathbf{E}_C(T_k, 1)^{-1} \kappa_{T_k}) = 0$$

contains a blowup sequence.

Our aim in sections 8,9,10 is to show that for any blowup sequence $\{T_k\}_{k \in \mathbb{N}}$ with $m \geq 1$ the associated harmonic blowups f_i, g_j are represented by two functions $f \in C^2(\mathbf{V} \cup \mathbf{L}), g \in C^2(\mathbf{W} \cup \mathbf{L})$ so that

$$\begin{aligned} f|_{\mathbf{V}} &= f_1 = f_2 = \dots = f_M, & g|_{\mathbf{W}} &= g_1 = \dots = g_m, \\ f|_{\mathbf{L}} &= 0 = g|_{\mathbf{L}}, & Df(0) &= Dg(0). \end{aligned}$$

8.2 Lemma

We make a slight modification to the proof of this lemma as presented in [14], to make clearer the application to 5.3.7 of [10]. This lemma will be used for example in Theorem 9.3 to show certain blowups have zero trace over \mathbf{L} . The proof requires referencing 5.3.7 of [10], which gives essentially the same result at the interior for minimizers of an elliptic integrand.

Lemma. *For every blowup sequence $\{T_k\}_{k \in \mathbb{N}}$ with associate blowups f_i, g_j , the two functions $\Pi, \Psi : B_1^n \rightarrow \mathbb{R}$ defined by*

$$\begin{aligned} \Pi|_{\mathbf{V}} &= \sum_{i=1}^M f_i, \quad \Pi|_{\mathbf{W}} = \sum_{j=1}^m g_j, \quad \Pi|_{\mathbf{L}} = 0, \\ \Psi|_{\mathbf{V}} &= \min\{|f_1|, |f_2|, \dots, |f_M|\}, \quad \Psi|_{\mathbf{W} \cup \mathbf{L}} = 0, \end{aligned}$$

both have locally square integrable weak gradients. Hence the function

$$\min\{|f_1|, |f_2|, \dots, |f_M|\} : \mathbf{V} \rightarrow \mathbb{R}$$

has zero trace on \mathbf{L} .

Proof. We take as in [14] and above

$$\varepsilon_k = \mathbf{E}_C(T_k, 1)^{\frac{1}{2}}, \quad \kappa_k = \kappa_{T_k}.$$

Define $q : \mathbb{R} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ by

$$q(t, x) = (x_1, \dots, x_{n-1}, tx_n, tx_{n+1})$$

for $(t, x) \in \mathbb{R} \times \mathbb{R}^{n+1}$ and subsequently define

$$Q_k = Q_{T_k} = q_{\#}((\mathbb{E}^1 \llcorner [0, 1]) \times (\partial T \llcorner C_2)),$$

as in Lemma A.0.5. Fix any $r \in (0, 1)$ and define

$$S_k = (T_k - Q_k + (M - m)(\mathbb{E}^n \llcorner \mathbf{W})) \llcorner C_r.$$

Now, the second identity of (A.0.7) implies

$$\mathbf{M}(\mathbf{p}_{\#} T_k \llcorner \bar{B}_r^n) \leq M \mathcal{H}^n(\mathbf{V} \cap \bar{B}_r^n) + m \mathcal{H}^n(\mathbf{W} \cap \bar{B}_r^n) + \mathbf{M}(\mathbf{p}_{\#} Q_k \llcorner \bar{B}_r^n).$$

Using this together with (A.0.7) and (A.0.6) we compute

$$\begin{aligned}
\partial S_k \llcorner (B_r^n \times \mathbb{R}) &= 0, \quad \mathbf{p}_\# S_k \llcorner B_r^n = M \mathbf{E}^n \llcorner B_r^n, \\
\mathbf{E}_C(S_k, r) &\leq r^{-n} \mathbf{M}(T_k \llcorner C_r) + r^{-n} \mathbf{M}(Q_k \llcorner C_1) + (M - m) r^{-n} \mathcal{H}^n(\mathbf{W} \cap \bar{B}_r^n) \\
&\quad - r^{-n} \mathbf{M}((M \mathbf{E}^n \llcorner \mathbf{V} + M \mathbf{E}^n \llcorner \mathbf{W}) \llcorner \bar{B}_r^n) \\
&= r^{-n} \mathbf{M}(T_k \llcorner C_r) + r^{-n} \mathbf{M}(Q_k \llcorner C_r) \\
&\quad - r^{-n} (M \mathcal{H}^n(\mathbf{V} \cap \bar{B}_r^n) + m \mathcal{H}^n(\mathbf{W} \cap \bar{B}_r^n)) \\
&\leq \mathbf{E}_C(T_k, r) + r^{-n} \mathbf{M}(Q_k \llcorner C_r) + r^{-n} \mathbf{M}(\mathbf{p}_\# Q_k \llcorner \bar{B}_r^n) \\
&\leq r^{-n} \varepsilon_k + (M - m) \kappa_k \varpi_{n-1} r^\alpha \left(\frac{\alpha}{2} \right) \left(1 + \frac{\alpha^2 \kappa_k^2}{4} r^{2\alpha} + \frac{\alpha^4 \kappa_k^4}{16} r^{4\alpha} \right)^{\frac{1}{2}} \\
&\quad + (M - m) \kappa_k \varpi_{n-1} r^\alpha.
\end{aligned}$$

We can hence apply Lemma 7.1 with V, S, \mathbf{m} replaced with B_r^n, S_k, M to obtain corresponding $S_{k,1}, \dots, S_{k,M}$. Observe that for each $i \in \{1, \dots, M\}$

$$\limsup_{k \rightarrow \infty} \varepsilon_k^{-2} \mathbf{E}_C(S_{k,i}, r) \leq \limsup_{k \rightarrow \infty} \varepsilon_k^{-2} \mathbf{E}_C(S_k, r) \leq r^{-n}$$

by Definition (8.1). Using (8.1.2), we see that the proof of 5.3.7 of [10] carries over with

$$\kappa, \lambda, \Psi_\nu, m, n, r_\nu, s_\nu, \varepsilon_\nu, \alpha$$

replaced by

$$1, 1, \text{the parameteric area integrand}, n, n+1, r, 1, \varepsilon_k, r^{-n}.$$

To see this, we make two observations: first, by (6.1.2) with $\sigma = (1 - r)$ and (3.3.3) together with 4.1.8 of [10] and Lemma 26.25 of [26], we have

$$\begin{aligned}
&\text{spt } S_k \\
&\subset \left\{ x \in C_r : |x_{n+1}| \leq \max \left\{ \frac{(c_{14}(1 + c_{15}))^{\frac{1}{2}}}{(1 - r)^{n+\frac{1}{2}}} (\varepsilon_k + \kappa_k^{\frac{1}{2}}), \left(\frac{\alpha}{2} \right) \kappa_k r^{1+\alpha} \right\} \right\}
\end{aligned}$$

which is contained in $\{x \in C_r : |x_{n+1}| < 1\}$, for all sufficiently large k ; second, note that in 5.3.7 of [10] the only terms which use the hypothesis

$$\lim_{\nu \rightarrow \infty} (\varepsilon_\nu^{-1} r_\nu + \varepsilon_\nu^{-1} s_\nu) = 0$$

are $M_{\nu,1}(t)$ and $M_{\nu,3}(t)$, both of which vanish because the area integrand is a constant coefficient integrand.

Choosing locally integrable functions $\Pi_i : B_r^n \rightarrow \mathbb{R}$ so that

$$\int \Pi_i \cdot \phi \, d\mathcal{H}^n = \lim_{k \rightarrow \infty} \varepsilon_k^{-1} S_{k,i}((\phi \circ \mathbf{p}) \mathbf{q}_1 dx_1 \wedge \dots \wedge dx_n)$$

for any $\phi \in C_c^\infty(B_r^n)$, we deduce that each Π_i has locally square integrable weak gradient. By Definition 8.1 and the choice of $S_{k,i}$, the set of values $\{\Pi_i(y), \dots, \Pi_M(y)\}$ coincides for Lebesgue almost all $y \in \mathbf{W} \cap B_r^n$ with the (unordered) set $\{0, g_1(y), \dots, g_m(y)\}$. After changing each Π_i on an \mathcal{H}^n -null set, we conclude that

$$\Pi|_{B_r^n} = \sum_{i=1}^M \Pi_i, \quad \Psi|_{B_r^n} = \min\{|\Pi_1|, \dots, |\Pi_M|\}.$$

Since this holds for all $r \in (0, 1)$, then both Π, Ψ have locally square integrable weak gradients over B_1^n .

The last part of the lemma follows from section 26 of [28]. \square

8.3 Lemma

As noted in [7], one crucial step in [14] is to obtain information about the height of currents in a blowup sequence from bounds on the harmonic blowups. Proving this depends on a well-known barrier argument, see Corollary 4.3 of [13].

Lemma. *Suppose $\{T_k\}_{k \in \mathbb{N}}$ is a blowup sequence with associated harmonic blowups f_i, g_j and let $\varepsilon_k = \mathbf{E}_C(T_k, 1)^{\frac{1}{2}}$. If $\sigma \in (0, 1/2)$, $z \in B_{1-2\sigma}^{n-1}$, and K is a compact subset of $\mathbf{p}^{-1}(B_\sigma^n(z))$, then*

$$\begin{aligned} \limsup_{k \rightarrow \infty} \sup_{K \cap \text{spt } T_k} \varepsilon_k^{-1} \mathbf{q}_1 &\leq \max\left\{ \sup_{\mathbf{V} \cap \partial B_\sigma^n(z)} f_M, \sup_{\mathbf{W} \cap \partial B_\sigma^n(z)} g_m, 0 \right\}, \\ \liminf_{k \rightarrow \infty} \inf_{K \cap \text{spt } T_k} \varepsilon_k^{-1} \mathbf{q}_1 &\geq \min\left\{ \inf_{\mathbf{V} \cap \partial B_\sigma^n(z)} f_1, \inf_{\mathbf{W} \cap \partial B_\sigma^n(z)} g_1, 0 \right\}. \end{aligned}$$

Proof. For each $k \in \mathbb{N}$, let $\kappa_k = \kappa_{T_k}$, $\sigma_k = \sigma_{T_k}$, and $v_i^{(k)}, w_j^{(k)}$ be as in sections 3.3, 7.4, 8.1. We may assume $2\sigma_k < \sigma$.

To prove the first inequality, we use (6.1.2) and Theorem 7.4 to choose functions $b_k \in C^2(\partial B_\sigma^n(z))$ so that

$$\begin{aligned} \max\{v_M^{(k)}(y), \kappa_k\} &\leq b_k(y) \leq \max\{v_M^{(k)}(y), \kappa_k\} + \frac{1}{k} \text{ for } y \in \mathbf{V}_{2\sigma_k} \cap \partial B_\sigma^n(z) \\ \max\{w_m^{(k)}(y), \kappa_k\} &\leq b_k(y) \leq \max\{w_m^{(k)}(y), \kappa_k\} + \frac{1}{k} \text{ for } y \in \mathbf{W}_{2\sigma_k} \cap \partial B_\sigma^n(z), \\ \max\left\{\sup_{\mathbf{p}^{-1}(y) \cap \text{spt } T_k} \mathbf{q}_1, \kappa_k\right\} &\leq b_k(y) \\ &\leq \frac{2(c_{14}(1 + c_{15}))^{\frac{1}{2}}}{\sigma^{n+\frac{1}{2}}}(\varepsilon_k + \kappa_k^{\frac{1}{2}}) \text{ for } y \in \partial B_\sigma^n(z). \end{aligned}$$

Next we solve the Dirichlet problem (see Theorem 16.9 of [12]) to obtain $u_k \in C^2(\bar{B}_\sigma^n(z))$ so that $u_k|_{\partial B_\sigma^n(z)} = b_k$ and u_k satisfies the minimal surface equation over $B_\sigma^n(z)$. The maximum principle (see section 3.6 of [12]) implies $u_k \geq \kappa_k$. Also, a well-known barrier argument (see Corollary 4.3 of [13]) can be used to show

$$(8.3.1) \quad x_{n+1} \leq u_k(\mathbf{p}(x)) \text{ for } x \in \mathbf{p}^{-1}(B_\sigma^n(z)) \cap \text{spt } T_k;$$

this is 6.3(1) of [14]. Using these two facts we conclude

$$\begin{aligned} \text{spt } \partial(T_k \llcorner \mathbf{p}^{-1}(B_\sigma^n(z))) &\subseteq (\mathbf{p}^{-1}(\bar{B}_\sigma^n(z)) \cap \text{spt } \partial T_k) \cup (\mathbf{p}^{-1}(\partial B_\sigma^n(z)) \cap \text{spt } T_k) \\ &\subset \mathbf{p}^{-1}(\bar{B}_\sigma^n(z)) \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} \leq u_k(\mathbf{p}(x))\}. \end{aligned}$$

Using Lemma 7.3 and (6.1.2) to obtain local interior gradient bounds on $\varepsilon_k^{-1}u_k$, independent of k , and using the interior Hölder estimates for the gradient (see Chapter 13 of [12]), we find a bounded harmonic function u on $B_\sigma^n(z)$ and an increasing sequence $\{k_l\}_{l \in \mathbb{N}}$ so that as $l \rightarrow \infty$

$$(8.3.2) \quad \varepsilon_{k_l}^{-1}u_{k_l} \rightarrow u \text{ uniformly on compact subsets of } B_\sigma^n(z);$$

this is as in 6.3(2) of [14]. Recalling (8.1.1)-(8.1.4) and using linear barriers, we readily verify that

$$\begin{aligned} \lim_{\tilde{y} \in B_\sigma^n(z), \tilde{y} \rightarrow y} u(\tilde{y}) &= \max\{f_M(y), 0\} \text{ for } y \in \mathbf{V} \cap \partial B_\sigma^n(z), \\ \lim_{\tilde{y} \in B_\sigma^n(z), \tilde{y} \rightarrow y} u(\tilde{y}) &= \max\{g_m(y), 0\} \text{ for } y \in \mathbf{W} \cap \partial B_\sigma^n(z). \end{aligned}$$

From the Poisson integral formula, we then obtain the inequality

$$\sup_{B_\sigma^n(z)} u \leq \max\left\{ \sup_{\mathbf{V} \cap \partial B_\sigma^n(z)} f_M, \sup_{\mathbf{W} \cap \partial B_\sigma^n(z)} g_m, 0 \right\},$$

which along with (8.3.1),(8.3.2) establishes the first inequality of the conclusion.

The second inequality similarly follows using a lower barrier. \square

8.4 Lemma

We here show that if the harmonic blowups f_i and g_j are each restrictions to \mathbf{V} and respectively \mathbf{W} of some multiple of \mathbf{q}_0 , then in fact f_i and g_j are the restriction to \mathbf{V} and respectively \mathbf{W} of $\beta \mathbf{q}_0$ for one value of $\beta \in \mathbb{R}$. We will use this lemma most notably in the proof of Theorem 11.1, to show that if T_k is a blowup sequence with linear harmonic blowups as described above, then for sufficiently large $k \in \mathbb{N}$, if we tilt T_k by β and rescale by $\tau = \tau(n, M, m, \alpha)$, then the new cylindrical excess is proportional to τ by a factor depending on $\mathbf{E}_C(T_k, 1)$ and κ_{T_k} .

In this section we need $m \geq 1$.

Lemma. *Suppose $m \geq 1$. If $\beta_1 \leq \beta_2 \leq \dots \leq \beta_M$ and $\gamma_1 \geq \dots \geq \gamma_m$, while $\{T_k\}_{k \in \mathbb{N}}$ is a blowup sequence with associated harmonic blowups*

$$\begin{aligned} f_i(y) &= \beta_i y_n \text{ for } y \in \mathbf{V}, \\ g_j(y) &= \gamma_j y_n \text{ for } y \in \mathbf{W}, \end{aligned}$$

then

$$\beta_1 = \beta_2 = \dots = \beta_M = \gamma_1 = \dots = \gamma_m,$$

and

$$\lim_{k \rightarrow \infty} \sup_{x \in C_\rho \cap \text{spt } T_k} |\varepsilon_k^{-1} x_{n+1} - \beta_1 x_n| = 0$$

for $\rho \in (0, 1)$, where $\varepsilon_k = \mathbf{E}_C(T_k, 1)^{\frac{1}{2}}$.

The proof proceeds partly by proving that the function $b : \bar{B}_{\frac{1}{2}}^n \rightarrow \mathbb{R}$ defined by $b(y) = \beta_M y_n$ for $y \in \bar{B}_{\frac{1}{2}}^n \cap \text{Clos } \mathbf{V}$ and $b(y) = \gamma_m y_n$ for $y \in \bar{B}_{\frac{1}{2}}^n \cap \text{Clos } \mathbf{W}$ is harmonic; b is the “top sheet” of the harmonic blowups. Showing this is done by using the area minimality of each T_k in the blowup sequence, to directly show b minimizes the Dirichlet integral. To do this, we must crucially use (A.0.8), which means we must assume $m \geq 1$.

Proof. While this lemma is analogous to Lemma 6.4 of [14], we must make some changes to the proof. In particular, we observe our use of Lemma 5.2, which differs from the corresponding Lemma 3.2 of [14].

Let $v_i^{(k)}, w_j^{(k)}$ be as in Definition 8.1, and let

$$\begin{aligned}\Lambda &= \max\{|\beta_1|, |\beta_M|, |\gamma_1|, |\gamma_m|\}, \\ \lambda &= \min\left(\{1\} \cup \{\beta_{i+1} - \beta_i : \beta_i < \beta_{i+1}\} \cup \{\gamma_j - \gamma_{j+1} : \gamma_j > \gamma_{j+1}\}\right).\end{aligned}$$

For each $\sigma \in (0, \min\{\frac{\lambda}{2}, \frac{1}{16}\})$, we use (7.4.1), (7.4.2), (8.1.1)-(8.1.4), and Lemma 8.3 to find $N_\sigma \in \mathbb{N}$ so that for $k \geq N_\sigma$

$$(8.4.1) \quad \sigma_{T_k} < \frac{\sigma}{4}, \quad \varepsilon_k^2 < \sigma, \quad \kappa_{T_k} < \sigma^3 \varepsilon_k^2$$

$$(8.4.2) \quad \sup_{\mathbf{V}_{\sigma/2}} |v_i^{(k)} - \varepsilon_k \beta_i \mathbf{q}_0|^2 \leq \sigma^{n+4} \varepsilon_k^2 \text{ for } i \in \{1, \dots, M\},$$

$$(8.4.3) \quad \sup_{\mathbf{W}_{\sigma/2}} |w_j^{(k)} - \varepsilon_k \gamma_j \mathbf{q}_0|^2 \leq \sigma^{n+4} \varepsilon_k^2 \text{ for } j \in \{1, \dots, m\},$$

$$(8.4.4) \quad \sup_{C_{\sigma+\frac{3}{4}} \cap (\text{spt } T_k) \setminus \mathbf{p}^{-1}(\mathbf{V}_{2\sigma} \cup \mathbf{W}_{2\sigma})} |\mathbf{q}_0| \leq 2\Lambda\sigma\varepsilon_k;$$

these are as in equations 6.4(1)-(4) of [14].

Next, if $k \geq N_\sigma$ then (7.4.1) and (8.4.1) imply $|Dv_i^{(k)}| \leq c_{25}$ for $y \in \mathbf{V}_{T_k}$. Using this, for each $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ we apply (7.2.3) with

$$\begin{aligned}u &= D_j(v^{(k)} - \varepsilon_k \beta_i \mathbf{q}_0) = D_j v_i^{(k)} - \varepsilon_k \beta_i \delta_{nj} \\ a_{kl} &= \frac{1}{\sqrt{1+|p|^2}} \left(\delta_{kl} - \frac{p_k p_l}{1+|p|^2} \right) \Big|_{p=Dv_i^{(k)}},\end{aligned}$$

then apply (7.2.2) with $u = v_i^{(k)} - \varepsilon_k \beta_i \mathbf{q}_0$ and

$$a_{kl} = \int_0^1 \frac{1}{\sqrt{1+|p|^2}} \left(\delta_{kl} - \frac{p_k p_l}{1+|p|^2} \right) \Big|_{p=tDv_i^{(k)}+(1-t)\varepsilon_k \beta_i e_n} dt$$

in order to conclude

$$\begin{aligned}
(8.4.5) \quad & \sup_{\mathbf{V}_\sigma} |D(v_i^{(k)} - \varepsilon_k \beta_i \mathbf{q}_0)| \\
& \leq \frac{c_{26}}{\sigma^n} \int_{\mathbf{V}_{\frac{\sigma}{2}}} |D(v_i^{(k)} - \varepsilon_k \beta_i \mathbf{q}_0)|^2 d\mathcal{H}^n \\
& \leq \frac{c_{27}}{\sigma^{n+2}} \int_{\mathbf{V}_{\frac{\sigma}{4}}} |v_i^{(k)} - \varepsilon_k \beta_i \mathbf{q}_0|^2 d\mathcal{H}^n \leq c_{27} \varpi_n \sigma^2 \varepsilon_k^2
\end{aligned}$$

with c_{26}, c_{27} depending in n, M, m ; this is 6.4(5) of [14]. We similarly verify

$$\sup_{\mathbf{W}_\sigma} |D(w_j^{(k)} - \varepsilon_k \gamma_j \mathbf{q}_0)|^2 \leq c_{27} \varpi_n \sigma^2 \varepsilon_k^2$$

for each $j \in \{1, \dots, m\}$.

Next, with $\sigma \in (0, \min\{\frac{\lambda}{2}, \frac{1}{16}\})$, $i \in \{1, \dots, M\}$, and $j \in \{1, \dots, m\}$ define

$$\begin{aligned}
H^\sigma &= \{x \in \mathbb{R}^{n+1} : |x_n| \leq \sigma\}, \\
I_{i,k}^\sigma &= \{x \in \mathbf{p}^{-1}(\mathbf{V}_\sigma) : |\varepsilon_k^{-1} x_{n+1} - \beta_i x_n| < \lambda \sigma / 2\}, \\
J_{j,k}^\sigma &= \{x \in \mathbf{p}^{-1}(\mathbf{W}_\sigma) : |\varepsilon_k^{-1} x_{n+1} - \gamma_j x_n| < \lambda \sigma / 2\};
\end{aligned}$$

note that

$$\begin{aligned}
I_{i,k}^\sigma \cap I_{\tilde{i},k}^\sigma &\neq \emptyset \text{ if and only if } \beta_i = \beta_{\tilde{i}}, \\
J_{j,k}^\sigma \cap J_{\tilde{j},k}^\sigma &\neq \emptyset \text{ if and only if } \gamma_j = \gamma_{\tilde{j}}.
\end{aligned}$$

For each $k \geq N_\sigma$ we also define the set

$$G_k^\sigma = H^\sigma \cup \bigcup_{i=1}^M I_{i,k}^\sigma \cup \bigcup_{j=1}^m J_{j,k}^\sigma,$$

and the Lipschitz map $\Lambda_k^\sigma : G_k^\sigma \rightarrow \mathbb{R}^{n+1}$ given by

$$\begin{aligned}
L_k^\sigma(x) &= \mathbf{p}(x) && \text{for } x \in H^\sigma, \\
L_k^\sigma(x) &= (\mathbf{p}(x), \varepsilon_k \beta_i (x_n - \sigma)) && \text{for } x \in I_{i,k}^\sigma, \\
L_k^\sigma(x) &= (\mathbf{p}(x), \varepsilon_k \gamma_j (x_n + \sigma)) && \text{for } x \in J_{j,k}^\sigma.
\end{aligned}$$

Finally, let $\phi \in C^1(\bar{B}_1^n; [0, 1])$ be a function with

$$\phi|_{\bar{B}_{1/2}^n} \equiv 0, \quad \phi|_{\bar{B}_1^n \setminus B_{3/4}^n} \equiv 1, \quad \sup |D\phi| \leq 5,$$

and define the Lipschitz mapping

$$F_k^\sigma : G_k^\sigma \cup (\mathbb{R}^{n+1} \setminus C_{\frac{3}{4}}) \rightarrow \mathbb{R}^{n+1},$$

by

$$\begin{aligned} F_k^\sigma(x) &= x && \text{for } x \in \mathbb{R}^{n+1} \setminus C_{\frac{3}{4}}, \\ F_k^\sigma(x) &= (1 - (\phi \circ \mathbf{p}))L_k^\sigma(x) + (\phi \circ \mathbf{p})(x) && \text{for } x \in G_k^\sigma. \end{aligned}$$

We wish to estimate $\mathbf{M}(F_{k\#}^\sigma T_k) - \mathbf{M}(T_k)$. First notice that

$$(8.4.6) \quad \mathbf{p}^{-1}(\mathbf{V}_\sigma) \cap \text{spt } T_k = \cup_{i=1}^M \text{graph}_{\mathbf{V}_\sigma} v_i^{(k)} \subset G_k^\sigma,$$

$$(8.4.7) \quad \mathbf{p}^{-1}(\mathbf{V}_\sigma) \cap F_{k\#}^\sigma \text{spt } T_k = \cup_{i=1}^M \text{graph}_{\mathbf{V}_\sigma} u_i^{(k)} \subset G_k^\sigma,$$

as in 6.4(6)(7) of [14], where we define

$$\begin{aligned} u_i^{(k)} &= (1 - \phi)\varepsilon_k \beta_i(\mathbf{q}_0 - \sigma) + \phi v_i^{(k)} \\ &= (1 - \phi)\varepsilon_k \beta_i \mathbf{q}_0 + \phi v_i^{(k)} - (1 - \phi)\varepsilon_k \beta_i \sigma \end{aligned}$$

over \mathbf{V}_σ ; hence,

$$(8.4.8) \quad \begin{aligned} \sup_{\mathbf{V}_\sigma} \left(|Du_i^{(k)}| + |Dv_i^{(k)}| \right) &\leq c_{28}(1 + \Lambda)\varepsilon_k, \\ u_i^{(k)} - v_i^{(k)} &= (1 - \phi)(\varepsilon_k \beta_i \mathbf{q}_0 - v_i^{(k)}) \\ &\quad - (1 - \phi)\varepsilon_k \beta_i \sigma \text{ over } \mathbf{V}_\sigma. \end{aligned}$$

$$(8.4.9) \quad \begin{aligned} &\sup_{\mathbf{V}_\sigma} \left| Du_i^{(k)} + Dv_i^{(k)} \right| \\ &\leq \sup_{\mathbf{V}_\sigma} \left| (1 - \phi)D(\varepsilon_k \beta_i \mathbf{q}_0 - v_i^{(k)}) + (\varepsilon_k \beta_i \mathbf{q}_0 - v_i^{(k)} + \varepsilon_k \beta_i \sigma)D\phi \right| \\ &\leq c_{29}(1 + \Lambda)\sigma\varepsilon_k \end{aligned}$$

by (8.4.2) and (8.4.5), with c_{28}, c_{29} depending on n, M, m ; these are as in 6.4(8)(9) of [14]. Using (8.4.6), (8.4.7), (8.4.8), and (8.4.9) we estimate

$$(8.4.10) \quad \begin{aligned} &\mathbf{M}(F_{k\#}^\sigma(T_k \llcorner \mathbf{p}^{-1}(\mathbf{V}_\sigma))) - \mathbf{M}(T_k \llcorner \mathbf{p}^{-1}(\mathbf{V}_\sigma)) \\ &= \sum_{i=1}^M \int_{\mathbf{V}_\sigma} \left((1 + |Du_i^{(k)}|^2)^{1/2} - (1 + |Dv_i^{(k)}|^2)^{1/2} \right) d\mathcal{H}^n \\ &\leq \sum_{i=1}^M \int_{\mathbf{V}_\sigma} |Du_i^{(k)} - Dv_i^{(k)}| \left(|Du_i^{(k)}| + |Dv_i^{(k)}| \right) d\mathcal{H}^n \\ &\leq c_{30}(1 + \Lambda)^2 \sigma \varepsilon_k^2 \end{aligned}$$

with c_{30} depending on n, M, m ; this is as in 6.4(10) of [14]. Similarly, we verify that

$$(8.4.11) \quad \mathbf{M}(F_{k\sharp}^\sigma(T_k \lrcorner \mathbf{p}^{-1}(\mathbf{W}_\sigma))) - \mathbf{M}(T_k \lrcorner \mathbf{p}^{-1}(\mathbf{W}_\sigma)) \leq c_{30}(1 + \Lambda)^2 \sigma \varepsilon_k^2$$

as in 6.4(11) of [14].

Next we infer from (3.2.1) that

$$\begin{aligned} \mu_{T_k}(H^{2\sigma} \cap C_{\frac{3}{4}+\sigma}) &\leq \mathbf{M}(\mathbf{p}_\sharp(T_k \lrcorner H^{2\sigma} \cap C_{\frac{3}{4}+\sigma})) + (3/4 + \sigma)^n \mathbf{E}_C(T, 3/4 + \sigma) \\ &\leq \mathbf{M}(\mathbf{p}_\sharp(T_k \lrcorner H^{2\sigma} \cap C_{\frac{3}{4}+\sigma})) + \varepsilon_k^2. \end{aligned}$$

Using (A.0.7) and (A.0.6) with $r = (\frac{3}{4} + \sigma)$ we compute

$$\begin{aligned} \mathbf{M}(\mathbf{p}_\sharp(T_k \lrcorner H^{2\sigma} \cap C_{\frac{3}{4}+\sigma})) &\leq (M + m) \varpi_{n-1}(3/4 + \sigma)^{n-1} (2\sigma) \\ &\quad + (M - m) \kappa_{T_k} \varpi_{n-1}(3/4 + \sigma)^{n+\alpha}. \end{aligned}$$

The previous two calculations, $\sigma \leq \frac{1}{16}$, and (8.4.1) imply that for $k \geq N_\sigma$

$$(8.4.12) \quad \mu_{T_k}(H^{2\sigma} \cap C_{\frac{3}{4}+\sigma}) \leq c_{31}(\sigma + \kappa_{T_k}) + \varepsilon_k^2 \leq 2c_{31}\sigma,$$

with c_{31} depending on n, M, m ; this is as in 6.4(12) of [14].

Noting that

$$F_k^\sigma(x) = (\mathbf{p}(x), \phi(\mathbf{p}(x))x_{n+1}) \text{ for all } x \in H^\sigma \cap C_{3/4},$$

then Lemma 5.2 (with $A = H^\sigma \cap C_{\frac{3}{4}}$ and $\tau = \sigma$) and (8.4.6), (8.4.12) give for c_{32}, c_{33} depending on n, M, m

$$\begin{aligned} (8.4.13) \quad &\mathbf{M}(F_{k\sharp}^\sigma(T_k \lrcorner H^\sigma)) - \mathbf{M}(T_k \lrcorner H^\sigma) \\ &= \mathbf{M}(F_{k\sharp}^\sigma(T_k \lrcorner H^\sigma)) - \mathbf{M}(T_k \lrcorner H^\sigma) \\ &\leq \frac{c_{32}}{\sigma^2} \left(\kappa_{T_k} + \int_{H^{2\sigma} \cap C_{\frac{3}{4}+\sigma}} \mathbf{q}_1^2 d\mu_{T_k} \right) \\ &\leq c_{33}(1 + \Lambda) \sigma \varepsilon_k^2, \end{aligned}$$

using as well $\kappa_{T_k} < \sigma^3 \varepsilon_k^2$ from (8.4.1); this is as in 6.4(13) of [14].

Combining (8.4.10), (8.4.11), and (8.4.13) gives the desired estimate

$$(8.4.14) \quad \mathbf{M}(F_{k\sharp}^\sigma T_k) - \mathbf{M}(T_k) \leq (2c_{30} + c_{33})(1 + \Lambda)^2 \sigma \varepsilon_k^2$$

for all $k \geq N_\sigma$.

We will use this estimate to show that the function $b : \bar{B}_{\frac{1}{2}}^n \rightarrow \mathbb{R}$

$$\begin{aligned} b(y) &= \beta_M y_n & \text{for } y \in \bar{B}_{\frac{1}{2}}^n \cap \text{Clos } \mathbf{V} \\ b(y) &= \gamma_m y_n & \text{for } y \in \bar{B}_{\frac{1}{2}}^n \cap \text{Clos } \mathbf{W}. \end{aligned}$$

is harmonic in $B_{\frac{1}{2}}^n$. For this purpose let $\zeta : \bar{B}_{\frac{1}{2}}^n \rightarrow \mathbb{R}$ be any Lipschitz function with $\zeta|_{\partial \bar{B}_{\frac{1}{2}}^n} = b|_{\partial \bar{B}_{\frac{1}{2}}^n}$, and let $\{\sigma_l\}_{l=1}^\infty$ be any decreasing sequence of positive numbers with limit zero and $\sigma_1 < \min\{\frac{\lambda}{2}, \frac{1}{16}\}$. For each $l \in \mathbb{N}$, let $k_l = N_{\sigma_l}$ and define $b_l : \bar{B}_{\frac{1}{2}}^n \rightarrow \mathbb{R}$ by

$$\begin{aligned} b_l(y) &= \beta_M(y_n - \sigma_l) & \text{for } y \in \bar{B}_{\frac{1}{2}}^n \cap \mathbf{V}_{\sigma_l} \\ b_l(y) &= \gamma_m(y_n - \sigma_l) & \text{for } y \in \bar{B}_{\frac{1}{2}}^n \cap \mathbf{W}_{\sigma_l} \\ b_l(y) &= 0 & \text{for } y \in \bar{B}_{\frac{1}{2}}^n \setminus (\mathbf{V}_{\sigma_l} \cup \mathbf{W}_{\sigma_l}), \end{aligned}$$

and choose $\zeta_l : \bar{B}_{\frac{1}{2}}^n \rightarrow \mathbb{R}$ Lipschitz so that $\zeta_l|_{\partial \bar{B}_{\frac{1}{2}}^n} = b_l|_{\partial \bar{B}_{\frac{1}{2}}^n}$,

$$\lim_{l \rightarrow \infty} \int_{\bar{B}_{\frac{1}{2}}^n} |D\zeta_l - D\zeta|^2 d\mathcal{H}^n = 0, \quad \limsup_{l \rightarrow \infty} \sup |D\zeta_l| < \infty.$$

Also, define

$$\begin{aligned} R_l &= (-1)^{n-1} \left(\partial(\mathbb{E}^{n+1} \llcorner \{x \in C_{\frac{1}{2}} : x_{n+1} > \varepsilon_{k_l} b_l(\mathbf{p}(x))\}) \right) \llcorner C_{\frac{1}{2}} \\ S_l &= (-1)^{n-1} \left(\partial(\mathbb{E}^{n+1} \llcorner \{x \in C_{\frac{1}{2}} : x_{n+1} > \varepsilon_{k_l} \zeta_l(\mathbf{p}(x))\}) \right) \llcorner C_{\frac{1}{2}}, \end{aligned}$$

(note the slight difference from the corresponding definitions in [14]). Using (8.4.7) with $\sigma = \sigma_l$, $k = k_l$ (as well as the analogous identity over \mathbf{W}_{σ_l}), (8.4.1), and (A.0.8) with $T = T_{k_l}$, $\sigma = \sigma_l$ we compute

$$(8.4.15) \quad \mathbf{M}(F_{k_l}^{\sigma_l}(T_{k_l} \llcorner C_1)) = \mathbf{M}(F_{k_l}^{\sigma_l}(T_{k_l} \llcorner C_1) - R_l) + \mathbf{M}(R_l)$$

because $F_{k_l}^{\sigma_l}|_{C_{\frac{1}{2}}} = L_{k_l}^{\sigma_l}|_{C_{\frac{1}{2}}}$; this is as in 6.4(15) of [14].

From the area minimality of T_{k_l} and the equations

$$\partial(T_{k_l} \llcorner C_1) = \partial F_{k_l}^{\sigma_l}(T_{k_l} \llcorner C_1) + \partial P_l, \quad \partial(R_l - S_l) = 0,$$

where

$$P_l = Q_{k_l} - F_{k_l \#}^{\sigma_l} Q_{k_l}$$

with Q_{k_l} as in the proof of Lemma 8.2 or Lemma A.0.5, we deduce that

$$(8.4.16) \quad \begin{aligned} \mathbf{M}(T_{k_l} \llcorner C_1) &\leq \mathbf{M}(F_{k_l \#}^{\sigma_l}(T_{k_l} \llcorner C_1) + P_l - R_l + S_l) \\ &\leq \mathbf{M}(F_{k_l \#}^{\sigma_l}(T_{k_l} \llcorner C_1) - R_l) + \mathbf{M}(P_l) + \mathbf{M}(S_l); \end{aligned}$$

this is as in 6.4(16) of [14]. Combining (8.4.14), (8.4.15), and (8.4.16) gives the inequality

$$(8.4.17) \quad \mathbf{M}(R_l) - \mathbf{M}(S_l) \leq \mathbf{M}(P_l) + (2c_{30} + c_{33})(1 + \Lambda)^2 \sigma_l \varepsilon_{k_l}^2,$$

as in 6.4(17) of [14]. Using the definition of $F_{k_l}^{\sigma_l}$, Lemma 26.25 of [26] or 4.1.14 of [10], and (A.0.6) with $r = \frac{3}{4}$ gives

$$\limsup_{l \rightarrow \infty} \varepsilon_{k_l}^{-2} \mathbf{M}(P_l) \leq \limsup_{l \rightarrow \infty} (1 + (\text{Lip}(F_{k_l}^{\sigma_l}))^n) \varepsilon_{k_l}^{-2} \mathbf{M}(Q_{k_l} \llcorner C_{\frac{3}{4}}) = 0$$

by (8.1.2). We deduce from (8.4.17) that

$$\begin{aligned} 0 &\geq \limsup_{k \rightarrow \infty} \varepsilon_{k_l}^{-2} (\mathbf{M}(R_l) - \mathbf{M}(S_l)) \\ &= \limsup_{l \rightarrow \infty} \varepsilon_{k_l}^{-2} \left(\int_{\bar{B}_{\frac{1}{2}}^n} \sqrt{1 + \varepsilon_{k_l}^2 |Db_l|^2} d\mathcal{H}^n - \int_{\bar{B}_{\frac{1}{2}}^n} \sqrt{1 + \varepsilon_{k_l}^2 |D\zeta_l|^2} d\mathcal{H}^n \right) \\ &= \limsup_{l \rightarrow \infty} \varepsilon_{k_l}^{-2} \int_{\bar{B}_{\frac{1}{2}}^n} \frac{|Db_l|^2 - |D\zeta_l|^2}{\sqrt{1 + \varepsilon_{k_l}^2 |Db_l|^2} + \sqrt{1 + \varepsilon_{k_l}^2 |D\zeta_l|^2}} d\mathcal{H}^n \\ &= \frac{1}{2} \int_{\bar{B}_{\frac{1}{2}}^n} |Db|^2 - |D\zeta|^2 d\mathcal{H}^n. \end{aligned}$$

Since this holds for all Lipschitz $\zeta : \bar{B}_{\frac{1}{2}}^n \rightarrow \mathbb{R}$ with $\zeta|_{\partial \bar{B}_{\frac{1}{2}}^n} = b$, then we conclude b minimizes the Dirichlet integral and so is a harmonic function. In particular, b is differentiable, and hence $\beta_M = \gamma_m$.

Since $\beta_1 \leq \beta_2 \leq \dots \leq \beta_M = \gamma_m \leq \dots \leq \gamma_1$, then we can use a similar argument to show $\beta_1 = \gamma_1$ to complete the proof of the first conclusion.

To prove the second conclusion, we assume $\sigma \in (0, \frac{(1-\rho)}{2})$ and deduce from (8.4.2) and (8.4.3) that

$$\limsup_{k \rightarrow \infty} \sup_{(\text{spt } T_k) \setminus H^{\sigma/2}} |\varepsilon_k^{-1} \mathbf{q}_1 - \beta_1 \mathbf{q}_0| = 0$$

and from Lemma 8.3 that

$$\limsup_{k \rightarrow \infty} \sup_{H^{\sigma/2} \cap C_\rho \cap \text{spt } T_k} |\varepsilon_k^{-1} \mathbf{q}_1 - \beta_1 \mathbf{q}_0| \leq \frac{|\beta_1| \sigma}{2} + \frac{|\beta_1| \sigma}{2},$$

and let $\sigma \rightarrow 0$. □

9 Comparison of spherical and cylindrical excess

Sections 9.1, 9.3, analogous to sections 7.1, 7.3 of [14], give bounds for the cylindrical excess (at smaller radii) in terms of the spherical excess, for $T \in \mathcal{T}$ with small cylindrical excess. We use Lemma 9.1 and Theorem 9.3 to prove C^2 boundary regularity for harmonic blowups in section 10. Section 9.2, analogous to section 7.2 of [14], gives a general lemma about homogeneous degree one harmonic functions over \mathbf{V} . We introduce c_{34}, \dots, c_{37} .

Theorem 9.3 requires $m \geq 1$, but otherwise we assume $m \in \{0, \dots, M-1\}$.

9.1 Lemma

Lemma. *There exists positive constants $c_{34} \geq 1 + c_{14}$, c_{35} , and c_{36} , all depending on n, M, m , so that if $T \in \mathcal{T}$,*

$$\begin{aligned} \mathbf{E}_C(T, 1) + \kappa_T &\leq c_{34}^{-1}, \\ \sup_{C_{\frac{1}{4}} \cap \text{spt } T} \mathbf{q}_1^2 &\leq c_{35}^{-1} \mathbf{E}_S(T, 1), \end{aligned}$$

then

$$\mathbf{E}_C(T, 1/3) \leq c_{36} (\mathbf{E}_S(T, 1) + \kappa_T).$$

We note that we correct a typo in the statement appearing in Lemma 7.1 of [14], where in the second inequality of the hypothesis we need $\mathbf{E}_S(T, 1)$ and not $\mathbf{E}_C(T, 1)$ as it appears in [14]. The proof follows using the first variation Lemma (6.1).

Proof. The calculations carry over exactly from section 7.1 of [14], although we make a clarification for the reader.

With c_4 as in (4.2.5) and c_{12}, c_{14}, c_{15} from Lemma 6.1, all now depending on n, M, m , we let

$$\begin{aligned} c_{34} &= 2^{2n+2}(1+c_4)(1+c_{14})(1+c_{15}) \\ c_{35} &= 3^{2n+8}(1+M\varpi_n)c_{12}, \\ c_{36} &= 4^{3n+6}(1+M\varpi_n+c_4)c_{12}, \end{aligned}$$

We now assume for contradiction that $T \in \mathcal{T}$ satisfies the hypothesis of the lemma, but that

$$\mathbf{E}_S(T, 1) + \kappa_T < c_{36}^{-1} \mathbf{E}_C(T, 1/3).$$

Using (6.1.2) with $\sigma = \frac{1}{2}$ and $\mathbf{E}_C(T, 1) + \kappa_T \leq c_{34}^{-1}$ we compute

$$\sup_{C_{\frac{1}{2}} \cap \text{spt } T} \mathbf{q}_1^2 \leq 2^{2n+1} c_{14} (1+c_{15}) (\mathbf{E}_C(T, 1) + \kappa_T) \leq \frac{1}{2},$$

hence

$$(9.1.1) \quad C_{\frac{1}{2}} \cap \text{spt } T \subset B_1;$$

this is equation 7.1(1) of [14].

Let $\kappa = \kappa_T$, $\varepsilon = \mathbf{E}_C(T, 1/3)^{\frac{1}{2}}$,

$$\chi(x) = \max \left\{ 0, \frac{x_{n+1}}{|x|} - \frac{4\varepsilon}{\sqrt{c_{35}}} - \kappa \right\} \text{ for } x \in \mathbb{R}^{n+1}$$

and for each $k \in \mathbb{N}$ let $\zeta_k \in C^1(\mathbb{R})$ be such that

$$\zeta_k(t) = \max\{0, t^{-n} - 1\}^{1+\frac{1}{k}} \text{ for } t \geq 1/4.$$

The assumptions $\sup_{C_{\frac{1}{4}} \cap \text{spt } T} \mathbf{q}_1^2 \leq c_{35}^{-1} \mathbf{E}_S(T, 1)$ and

$\mathbf{E}_S(T, 1) + \kappa_T \leq c_{36}^{-1} \mathbf{E}_C(T, 1/3)$ then imply

$$\sup_{x \in (\text{spt } T) \cap C_{\frac{1}{4}} \setminus C_{\frac{1}{8}}} \frac{x_{n+1}}{|x|} \leq 8c_{35}^{-\frac{1}{2}} \mathbf{E}_S(T, 1)^{\frac{1}{2}} \leq \frac{8\varepsilon}{\sqrt{c_{35}c_{36}}}.$$

We can therefore choose a vector field $X_k \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ so that

$$\begin{aligned} X_k(x) &= 0 & \text{for } x \in \bar{B}_{\frac{1}{4}} \cap \text{spt } T \\ X_k(x) &= \chi^2(x) \zeta_k(|x|) x & \text{for } x \in \mathbb{R}^{n+1} \setminus \bar{B}_{\frac{1}{4}}. \end{aligned}$$

As in [14], X_k vanishes on

$$(\text{spt } \partial T) \cup (\bar{B}_{\frac{1}{4}} \cap \text{spt } T) \cup \left\{ x \in \mathbb{R}^{n+1} : x_{n+1} \leq |x| \left(\frac{4\varepsilon}{\sqrt{c_{35}}} + \kappa \right) \right\}.$$

Using the first variation formula Lemma 4.1 we compute

$$\begin{aligned} 0 &= \int \text{div}_T X_k \, d\mu_T \\ &= \int_{B_1 \setminus \bar{B}_{\frac{1}{4}}} \chi^2(x) \left(|x| \zeta'_k(|x|) \left(1 - \left(\frac{x}{|x|} \cdot \mathcal{V}^T \right)^2 \right) - n \zeta_k(|x|) \right) d\mu_T(x) \\ &\quad + \int_{B_1 \setminus \bar{B}_{\frac{1}{4}}} 2\chi(x) \zeta_k(|x|) x \cdot \nabla^T \left(\frac{x_{n+1}}{|x|} \right) d\mu_T(x). \end{aligned}$$

Computing as well

$$\left| x \cdot \nabla^T \left(\frac{x_{n+1}}{|x|} \right) \right| = \left| \frac{x_{n+1}}{|x|} \left(\frac{x}{|x|} \cdot \mathcal{V}^T \right)^2 - \mathcal{V}_{n+1}^T \left(\frac{x}{|x|} \cdot \mathcal{V}^T \right) \right| \leq 2 \left| \frac{x}{|x|} \cdot \mathcal{V}^T \right|,$$

we let $k \rightarrow \infty$ and deduce that

$$\begin{aligned} \frac{n}{2^n} \int_{B_1 \setminus \bar{B}_{\frac{1}{4}}} \chi^2 \, d\mu_T &\leq n \int_{B_1 \setminus \bar{B}_{\frac{1}{4}}} \frac{\chi^2(x)}{|x|^n} \left(\frac{x}{|x|} \cdot \mathcal{V}^T \right)^2 d\mu_T(x) \\ &\quad + 4 \int_{B_1 \setminus \bar{B}_{\frac{1}{4}}} \frac{\chi(x)}{|x|^n} \left| \frac{x}{|x|} \cdot \mathcal{V}^T \right| d\mu_T(x) \\ &\leq 5n4^n \int_{B_1 \setminus \bar{B}_{\frac{1}{4}}} \chi(x) \left| \frac{x}{|x|} \cdot \mathcal{V}^T \right| d\mu_T(x). \end{aligned}$$

Similarly, we verify that

$$\begin{aligned} \int_{B_1 \setminus \bar{B}_{\frac{1}{4}}} \min \left\{ 0, \frac{x_{n+1}}{|x|} + \frac{4\varepsilon}{\sqrt{c_{35}}} + \kappa \right\}^2 d\mu_T(x) \\ \leq 4^{2n+3} \int_{B_1 \setminus \bar{B}_{\frac{1}{4}}} \left(\frac{x}{|x|} \cdot \mathcal{V}^T \right)^2 d\mu_T(x). \end{aligned}$$

Adding these inequality we infer that

$$\begin{aligned}
\int_{B_1 \setminus \bar{B}_{\frac{1}{4}}} \mathbf{q}_1^2 d\mu_T &\leq \int_{B_1 \setminus \bar{B}_{\frac{1}{4}}} \frac{x_{n+1}^2}{|x|^2} d\mu_T(x) \\
&\leq 4 \left(\frac{4\varepsilon}{\sqrt{c_{35}}} + \kappa^2 \right) \mu_T(B_1 \setminus \bar{B}_{\frac{1}{4}}) \\
&\quad + 4^{2n+4} \int_{B_1 \setminus \bar{B}_{\frac{1}{4}}} \left(\frac{x}{|x|} \cdot \mathcal{V}^T \right)^2 d\mu_T(x).
\end{aligned}$$

We also compute, using $\sup_{C_{\frac{1}{4}} \cap \text{spt } T} \mathbf{q}_1^2 \leq c_{35}^{-1} \mathbf{E}_S(T, 1)$ and $\mathbf{E}_S(T, 1) + \kappa_T \leq c_{36}^{-1} \mathbf{E}_C(T, 1/3)$, that

$$\int_{\bar{B}_{\frac{1}{4}}} \mathbf{q}_1^2 d\mu_T \leq (c_{35}^{-1} c_{36}^{-1} \varepsilon^2) \mu_T(\bar{B}_{\frac{1}{4}}) \leq 16 c_{35}^{-1} \varepsilon^2 \mu_T(\bar{B}_{\frac{1}{4}}).$$

Using (4.3.1), (6.1.1) (with T, σ replaced by $(\boldsymbol{\eta}_{\frac{1}{3}\#} T) \llcorner B_3, 1/2$), (9.1.1), and (4.2.5), we conclude

$$\begin{aligned}
\varepsilon^2 &\leq 3^{n+2} c_{12} \left(\kappa + \int_{C_{\frac{1}{2}}} \mathbf{q}_1^2 d\mu_T \right) \\
&\leq 3^{n+2} c_{12} \left(\kappa + \int_{B_1} \mathbf{q}_1^2 d\mu_T \right) \\
&\leq 3^{n+2} c_{12} \left(\kappa + 16 c_{35}^{-1} \varepsilon^2 \mu_T(\bar{B}_{\frac{1}{4}}) + 64 c_{35}^{-1} \varepsilon^2 \mu_T(B_1 \setminus \bar{B}_{\frac{1}{4}}) \right. \\
&\quad \left. + 4 \kappa^2 \mu_T(B_1 \setminus \bar{B}_{\frac{1}{4}}) + 4^{2n+4} \int_{B_1 \setminus \bar{B}_{\frac{1}{4}}} \left(\frac{x}{|x|} \cdot \mathcal{V}^T \right)^2 d\mu_T(x) \right) \\
&\leq 3^{2n+4} c_{12} (1 + M \varpi_n) (16 c_{35}^{-1} \varepsilon^2 + \kappa) + 4^{3n+6} c_{12} (\mathbf{E}_S(T, 1) + c_4 \kappa) \\
&\leq (\varepsilon^2/2) + \frac{c_{36}}{2} (\mathbf{E}_S(T, 1) + \kappa)
\end{aligned}$$

which is a contradiction. \square

9.2 Remark

As in section 7.2 of [14], we make a general observation about homogeneous degree one harmonic functions over \mathbf{V} . We use this in the proof of Theorem 9.3 in order to apply Lemma 8.4.

Suppose $h : \mathbf{V} \rightarrow \mathbb{R}$ is harmonic and $h(\rho y) = \rho h(y)$ for all $\rho \in (0, 1)$ and $y \in \mathbf{V}$.

(9.2.1) If h is nonnegative, then h has zero trace
(see section 26 of [28]) on \mathbf{L} .

(9.2.2) If h has zero trace on \mathbf{L} , then $h = \beta \mathbf{q}_0 \mathbf{L} \mathbf{V}$ for some $\beta \in \mathbb{R}$.

These are exactly as in 7.2(1)(2) of [14].

Proof. To verify (9.2.1), note that on the open hemisphere

$$S_+^{n-1} = \{y \in \mathbb{R}^n : |y| = 1, y_n > 0\}$$

the spherical Laplacian has minimum eigenvalue $(n - 1)$. Letting $h_2(y) = 2h(y/2)$ for $y \in S_+^{n-1}$ and choosing $\lambda > 0$ so that the spherical domain $\Omega = \{y \in S_+^{n-1} : \lambda y_n > h_2(y)\}$ is nonempty, we observe that $(\lambda \mathbf{q}_0 - h_2)|_\Omega$ is a positive eigenfunction for the spherical Laplacian on Ω with eigenvalue $(n - 1)$. If h did not have zero trace on \mathbf{L} , then $S_+^{n-1} \setminus \Omega$ would have nonempty interior, which would contradict the strict monotonicity of the minimum eigenvalue of the spherical Laplacian with respect to the domain (see, for example, section 2.3 of [5]).

Statement (9.2.2) follows from the weak version of the Schwarz reflection principle. \square

9.3 Theorem

We conclude here as in Theorem 7.3 of [14], although with c_{37} with n, M, m . For this, we require $m \geq 1$, as we apply Theorem 8.4.

Theorem. Suppose $m \geq 1$. There is a positive constant c_{37} depending on n, M, m so that if $T \in \mathcal{T}$,

$$\begin{aligned} \mathbf{E}_C(T, 1) + \kappa_T &\leq (2c_{34})^{-1} \text{ where } c_{34} \text{ is as in Lemma 9.1,} \\ \mathbf{E}_C(T, 1/3) + \mathbf{E}_C(T, 1/3)^{-1} \kappa_T &\leq c_{37}^{-1}, \text{ and} \\ \mathbf{E}_C(T, 1/4) &\leq 2\mathbf{E}_C(\mathbf{rot}_{\theta\#} T, 1/4) \text{ whenever } \theta \in (-1/8, 1/8), \end{aligned}$$

then

$$\mathbf{E}_C(T, 1/4) \leq c_{37}(\mathbf{E}_S(T, 1) + \kappa_T).$$

Proof. There are only minor changes to the proof from section 7.3 of [14].

Take for contradiction a sequence $\{T_k\}_{k \in \mathbb{N}}$ in \mathcal{T} so that for each $k \in \mathbb{N}$

$$(9.3.1) \quad \begin{aligned} \mathbf{E}_C(T_k, 1) + \kappa_{T_k} &\leq (2c_{34})^{-1} \\ \mathbf{E}_C(T_k, 1/4) &\leq 2\mathbf{E}_C(\mathbf{rot}_{\theta\#}T_k, 1/4) \text{ whenever } \theta \in (-1/8, 1/8), \end{aligned}$$

and so that as $k \rightarrow \infty$

$$(9.3.2) \quad \begin{aligned} \mathbf{E}(T_k, 1/3) + \mathbf{E}_C(T_k, 1/3)^{-1} \kappa_{T_k} &\rightarrow 0 \\ \mathbf{E}_C(T_k, 1/4)^{-1} (\mathbf{E}_S(T_k, 1) + \kappa_{T_k}) &\rightarrow 0; \end{aligned}$$

these are as 7.3(1)(2) of [14]. Letting $S_k = (\boldsymbol{\eta}_{\frac{1}{3}\#}T_k)\mathbf{L}B_3$, then Remark (4.3) implies $S_k \in \mathcal{T}$. With

$$\varepsilon_k = \mathbf{E}_C(S_k, 1)^{\frac{1}{2}} = \mathbf{E}_C(T_k, 1/3)^{\frac{1}{2}}, \quad \kappa_k = \kappa_{S_k} \leq \frac{\kappa_{T_k}}{3}$$

we can by Lemma 8.1 assume $\{S_k\}_{k \in \mathbb{N}}$ is a blowup sequence with associated harmonic blowups f_i, g_j . We can compute using (3.2.1) and (4.2.1)

$$\begin{aligned} \limsup_{k \rightarrow \infty} \varepsilon_k^{-2} (\mathbf{E}_S(S_k, 1) + c_4 \kappa_k) \\ \leq \limsup_{k \rightarrow \infty} (4/3)^n \mathbf{E}_C(T_k, 1/4)^{-1} \left(e^{c_1 \kappa_k} \mathbf{E}_S(T_k, 1) \right. \\ \left. + (e^{c_1 \kappa_k} - 1) \left(\frac{M+m}{2} \right) \varpi_n + c_4 \kappa_k \right) = 0. \end{aligned}$$

Thus, we may apply (7.4.3) (with T replaced by S_k) and Definition 8.1 (with T_k replaced by S_k) to deduce that for each $i \in \{1, \dots, M\}$ and $j \in \{1, \dots, m\}$

$$\int_{\mathbf{V}} \left(\frac{\partial}{\partial r} \left(\frac{f_i(y)}{|y|} \right) \right)^2 |y|^{2-n} d\mathcal{H}^n(y) = 0 = \int_{\mathbf{W}} \left(\frac{\partial}{\partial r} \left(\frac{g_j(y)}{|y|} \right) \right)^2 |y|^{2-n} d\mathcal{H}^n(y),$$

so that for all $\rho \in (0, 1)$

$$f_i(\rho y) = \rho f_i(y) \text{ for } y \in \mathbf{V}, \quad g_j(\rho y) = \rho g_j(y) \text{ for } y \in \mathbf{W}.$$

By (9.2.1) the nonnegative function $f_M - f_1$ has zero trace on \mathbf{L} . Thus by Lemma 8.2, each

$$\begin{aligned} |f_i| &= (|f_i| - \min\{|f_1|, \dots, |f_M|\}) + \min\{|f_1|, \dots, |f_M|\} \\ &\leq (f_M - f_1) + \min\{|f_1|, \dots, |f_M|\} \end{aligned}$$

has zero trace on \mathbf{L} for each $i \in \{1, \dots, M\}$. By (9.2.2) we conclude $f_i = \beta_i \mathbf{q}_0|_{\mathbf{V}}$ for some $\beta_i \in \mathbb{R}$. By Lemma 8.2 we as well conclude $\sum_{j=1}^m g_j$ has zero trace on \mathbf{L} because $\sum_{i=1}^M f_i$ does. From (9.2.1) it follows that each function

$$\begin{aligned} m|g_j| &= \left| \sum_{\tilde{j}=1}^m (g_j - g_{\tilde{j}}) + \sum_{\tilde{j}=1}^m g_{\tilde{j}} \right| \\ &\leq m(g_m - g_1) + \left| \sum_{\tilde{j}=1}^m g_{\tilde{j}} \right| \end{aligned}$$

has zero trace on \mathbf{L} ; hence $g_j = \gamma_j \mathbf{q}_0|_{\mathbf{W}}$ for some $\gamma_j \in \mathbb{R}$ by (9.2.1).

From Lemma 8.4 (since $m \geq 1$) we conclude that

$$\beta_1 = \beta_2 = \dots = \beta_M = \gamma_1 = \dots = \gamma_m,$$

$$(9.3.3) \quad \lim_{k \rightarrow \infty} \sup_{C_{\frac{7}{8}} \cap \text{spt } S_k} |\varepsilon_k^{-1} \mathbf{q}_1 - \beta_1 \mathbf{q}_0| = 0;$$

this is 7.3(4) of [14] (note that there is no equation 7.3(3) in [14], as it was mistakenly skipped).

From Lemma 9.1, (9.3.2), (9.3.3), and the inequalities

$$\begin{aligned} \sup_{C_{\frac{1}{4}} \cap \text{spt } T_k} \mathbf{q}_1^2 &\leq \sup_{C_{\frac{7}{8}} \cap \text{spt } S_k} \mathbf{q}_1^2, \\ (3/4)^n \mathbf{E}_C(T_k, 1/4) &\leq \mathbf{E}_C(T_k, 1/3) = \varepsilon_k^2 \leq 3^n \mathbf{E}_C(T_k, 1), \end{aligned}$$

we deduce that $\beta_1 \neq 0$. Letting $\theta_k = \arctan(\beta_1 \varepsilon_k)$, we infer from Remark 6.2

$$R_k = (\boldsymbol{\eta}_{\frac{1}{4}\#} \mathbf{rot}_{\theta_k\#} T) \llcorner B_3 \in \mathcal{T}$$

for all $k \in \mathbb{N}$ sufficiently large, and from (9.3.3) and (6.1.1) that

$$\begin{aligned} (9.3.4) \quad &\limsup_{k \rightarrow \infty} \varepsilon_k^{-2} \mathbf{E}_C(\mathbf{rot}_{\theta_k\#} T_k, 1/4) \\ &= \limsup_{k \rightarrow \infty} \varepsilon_k^{-2} \mathbf{E}_C(T_k, 1) \\ &\leq \limsup_{k \rightarrow \infty} (c_{12} c_{13} \sup_{C_{\frac{7}{8}} \cap \text{spt } R_k} \varepsilon_k^{-2} \mathbf{q}_1^2 + c_{12} \varepsilon_k^{-2} \kappa_k) = 0; \end{aligned}$$

this is as in 7.3(5) of [14]. On the other hand, by (6.1.2) and (9.3.3),

$$\begin{aligned}
(9.3.5) \quad & \liminf_{k \rightarrow \infty} \varepsilon_k^{-2} \mathbf{E}_C(T_k, 1/4) \\
& \geq \liminf_{k \rightarrow \infty} \left((8^{2n+1} c_{14} c_{15})^{-1} \sup_{C_{\frac{1}{8}} \cap T_k} \varepsilon_k^{-2} \mathbf{q}_1^2 - (8^{n+1} c_{15})^{-1} \varepsilon_k^{-2} \kappa_k \right) \\
& = (8^{2n+1} c_{14} c_{15})^{-1} (\beta_1/8)^2 > 0;
\end{aligned}$$

this is as in 7.3(6) of [14]. Statements (9.3.4) and (9.3.5) now contradict (9.3.1) for k sufficiently large. \square

10 Boundary regularity of harmonic blowups

We remark as in [7], that a well-known key step in various regularity proofs for minimizing currents is to derive a decay rate $\mathbf{E}_C(\mathbf{rot}_{\vartheta} T, \rho) \approx \rho^\alpha$ as $\rho \searrow 0$ for $T \in \mathcal{T}$ with $\mathbf{E}_C(T, 1) + \kappa_T$ sufficiently small for a suitable angle ϑ ; this is done in Theorem 11.2. However, proving this decay rate for the cylindrical excess requires we prove all harmonic blowups corresponding to all blowup sequences are given by C^2 functions over $\mathbf{V} \cup \mathbf{L}$ and $\mathbf{W} \cup \mathbf{L}$; this is Theorem 10.2. We only need to make a clarification to the end of the proof of Lemma 10.1, which is as in Lemma 8.1 of [14]. We introduce c_{38}, \dots, c_{43} now depending on n, M, m .

We assume throughout this section that $m \geq 1$.

10.1 Lemma

To proof of Theorem 10.2 relies on an application of the Hopf boundary point lemma. For this, we must first show that all harmonic blowups are roughly differentiable at the origin, in the sense of the following lemma.

Lemma. *Suppose $m \geq 1$. If $\{T_k\}_{k \in \mathbb{N}}$ is a blowup sequence in \mathcal{T} with associated harmonic blowups f_i, g_j then*

$$\sup_{y \in \mathbf{V} \cap B_\rho^n} \frac{|f_i(y)|}{|y|} < \infty, \quad \sup_{y \in \mathbf{W} \cap B_\rho^n} \frac{|g_j(y)|}{|y|} < \infty,$$

for each $\rho \in (0, 1)$, $i \in \{1, \dots, M\}$, and $j \in \{1, \dots, m\}$.

Proof. For each $k \in \mathbb{N}$, letting $\varepsilon_k = \mathbf{E}_C(T_k, 1)^{\frac{1}{2}}$ and $\kappa_k = \kappa_{T_k}$, choose for each $\sigma \in (0, 1/12]$ a $\theta(k, \sigma) \in [-1/8, 1/8]$ so that

$$(10.1.1) \quad \mathbf{E}_C(\mathbf{rot}_{\theta(k, \sigma)\#} T, \sigma/4) \leq 2\mathbf{E}_C(\mathbf{rot}_{\theta\#} T, \sigma/4) \text{ whenever } |\theta| \leq 1/8;$$

this is as in 8.1(1) of [14]. From (6.1.1), (6.1.2), (6.2.3), (3.2.1), and (8.1.1), (8.1.2), we infer that

$$(10.1.2) \quad \theta(k, \sigma) \rightarrow 0 \text{ and } \mathbf{E}_C(\mathbf{rot}_{\theta(k, \sigma)\#} T_k, \sigma) \rightarrow 0 \text{ as } k \rightarrow \infty;$$

this is as in 8.1(2) of [14].

We now consider two possibilities:

Case 1. $\mathbf{E}_C(\mathbf{rot}_{\theta(k, \sigma)\#} T_k, \sigma/3) < \varepsilon_k^2$ for infinitely many $k \in \mathbb{N}$.

Case 2. $\mathbf{E}_C(\mathbf{rot}_{\theta(k, \sigma)\#} T_k, \sigma/3) \geq \varepsilon_k^2$ for all sufficiently large $k \in \mathbb{N}$.

In *Case 2*, using (10.1.1), (10.1.2), (6.2.3), (6.2.4), and (8.1.1), (8.1.2), (8.1.3), (8.1.4) we can choose $N_\sigma \in \mathbb{N}$ so that, for all $k \geq N_\sigma$, we have $\kappa_k \leq \varepsilon_k^2$,

$$\begin{aligned} S_k &= (\mathbf{rot}_{\theta(k, \sigma)\#} \eta_{\sigma\#} T_k) \lfloor B_3 \in \mathcal{T} \\ \mathbf{E}_C(S_k, 1) + \kappa_{S_k} &\leq (4\sigma)^{-n} \mathbf{E}_C((\eta_{\frac{1}{4}\#} \mathbf{rot}_{\theta(k, \sigma)\#} T_k) \lfloor B_3, 1) + \sigma^\alpha \kappa_k \\ &\leq (2c_{34})^{-1} \\ \mathbf{E}_C(S_k, 1/3) + \mathbf{E}_C(S_k, 1/3)^{-1} \kappa_{S_k} &\leq \left(\frac{3}{4\sigma}\right)^n \mathbf{E}_C((\eta_{\frac{1}{4}\#} \mathbf{rot}_{\theta(k, \sigma)\#} T_k) \lfloor B_3, 1) \\ &\quad + \varepsilon_k^{-2} \sigma^\alpha \kappa_k \leq c_{37}^{-1}, \\ \mathbf{E}_C(S_k, 1/4) &\leq 2\mathbf{E}_C(\mathbf{rot}_{\theta\#} S_k, 1/4) \text{ whenever } |\theta| \leq 1/8. \end{aligned}$$

These are exactly the assumptions needed to apply Theorem 7.3 with $T = S_k$ for $k \geq N_\sigma$, and so we conclude by (4.3.2), (3.3.5), and (8.1.2)

$$\begin{aligned} \mathbf{E}_C(\mathbf{rot}_{\theta(k, \sigma)\#}, \sigma/4) &= \mathbf{E}_C(S_k, 1/4) \\ &\leq c_{37}(\mathbf{E}_S(S_k, 1) + \kappa_{S_k}) \\ &\leq c_{37}(\mathbf{E}_S(T_k, \sigma) + \sigma^\alpha \kappa_k) \\ &\leq c_{38}(\mathbf{E}_S(T_k, 1) + \kappa_k) \\ &\leq c_{39} \varepsilon_k^2. \end{aligned}$$

Here, c_{38} depends on n, M, m as we used

$$\mathbf{E}_S(T_k, \sigma) \leq e^{c_{1\kappa_k}} \mathbf{E}_S(T_k, 1) + (e^{c_{1\kappa_k}} - 1) \left(\frac{M+m}{2} \right) \varpi_n$$

by (4.2.1); thus, c_{39} also depends on n, M, m .

If instead *Case 1* occurs so that $\mathbf{E}_C(\mathbf{rot}_{\theta(k,\sigma)\sharp} T_k, \sigma/3) < \varepsilon_k^2$ for infinitely many $k \in \mathbb{N}$, then we use (3.2.1) to conclude that in either *Case 1* or *Case 2*

$$\mathbf{E}_C(\mathbf{rot}_{\theta(k,\sigma)\sharp} T_k, \sigma/4) \leq c_{40} \varepsilon_k^2$$

for infinitely many $k \in \mathbb{N}$, where c_{40} depends on n, M, m . Since $S_k \in \mathcal{T}$, then we can apply (4.3.1), (6.1.2) with T, σ replaced by $(\mathbf{rot}_{\theta(k,\sigma)\sharp} \boldsymbol{\eta}_{\frac{\sigma}{4}\sharp} T_k) \llcorner B_3, 1/20$, as well as (8.1.2) to conclude

$$(10.1.3) \quad \sup_{C_{\frac{\sigma}{5}} \cap \text{spt } \mathbf{rot}_{\theta(k,\sigma)\sharp} T_k} |\mathbf{q}_1| \leq c_{41} \varepsilon_k \sigma$$

for infinitely many $k \in \mathbb{N}$, with c_{41} now depending on n, M, m ; this is as in 8.1(3) of [14].

Letting $\bar{y} = (y_1, \dots, y_{n-1}, -y_n)$ for $y \in \mathbb{R}^n$ and letting $v_i^{(k)}, w_j^{(k)}$ be associated with T_k as in Definition 8.1, we note that $\theta(k, \sigma) \rightarrow 0$ as $k \rightarrow \infty$, and we use (10.1.3) to estimate, for each $\tau \in (0, 1)$,

$$\begin{aligned} |v_i^{(k)}(y) - v_{\tilde{i}}^{(k)}(y)| &\leq 2c_{41} \varepsilon_k \sigma & \text{for } y \in \mathbf{V}_\tau \cap B_{\frac{\sigma}{5}}^n, \\ |w_j^{(k)}(y) - w_{\tilde{j}}^{(k)}(y)| &\leq 2c_{41} \varepsilon_k \sigma & \text{for } y \in \mathbf{W}_\tau \cap B_{\frac{\sigma}{5}}^n, \\ |v_i^{(k)}(y) + w_j^{(k)}(\bar{y})| &\leq 2c_{41} \varepsilon_k \sigma & \text{for } y \in \mathbf{V}_\tau \cap B_{\frac{\sigma}{5}}^n, \end{aligned}$$

for infinitely many $k \in \mathbb{N}$ and for all $\{i, \tilde{i}\} \subseteq \{1, \dots, M\}$ and all $\{j, \tilde{j}\} \subseteq \{1, \dots, m\}$.

From the arbitrariness of σ and (8.1.3), (8.1.4) we infer that

$$(10.1.4) \quad \begin{aligned} |f_i(y) - f_{\tilde{i}}(y)| &\leq 8c_{41} |y| & \text{for } y \in \mathbf{V} \cap B_{\frac{1}{60}}^n, \\ |g_j(y) - g_{\tilde{j}}(y)| &\leq 8c_{41} |y| & \text{for } y \in \mathbf{W} \cap B_{\frac{1}{60}}^n, \\ |f_i(y) + g_j(\bar{y})| &\leq 8c_{41} |y| & \text{for } y \in \mathbf{V} \cap B_{\frac{1}{60}}^n, \end{aligned}$$

for all $\{i, \tilde{i}\} \subseteq \{1, \dots, M\}$ and all $\{j, \tilde{j}\} \subseteq \{1, \dots, m\}$; this is 8.1(4) of [14].

We may also apply Lemma 8.2 to see that the functions

$$\begin{aligned}\Pi : B_1^n &\rightarrow \mathbb{R}, & \mathcal{P} : B_1^n &\rightarrow \mathbb{R} \\ \Pi|_{\mathbf{V}} &= \sum_{i=1}^M f_i, & \Pi|_{\mathbf{W}} &= \sum_{j=1}^m g_j, \quad \Pi|_{\mathbf{L}} = 0, \\ \mathcal{P}(y) &= \Pi(y) - \Pi(\bar{y}) \quad \text{for } y \in B_1^n\end{aligned}$$

have locally square integrable weak gradients. Since \mathcal{P} is odd in the second variable and $\mathcal{P}|_{\mathbf{V} \cup \mathbf{W}}$ is harmonic, \mathcal{P} is, by the weak version of the Schwarz reflection principle, harmonic on all of B_1^n ; hence, for each $\rho \in (0, 1)$

$$(10.1.5) \quad \sup_{y \in B_\rho^n} \frac{|\mathcal{P}(y)|}{|y|} < \infty;$$

this is 8.1(5) of [14].

On the other hand, for $y \in \mathbf{V}$

$$\begin{aligned}\mathcal{P}(y) &= \sum_{i=1}^M f_i(y) - \sum_{j=1}^m g_j(\bar{y}) \\ &= \sum_{i=1}^M f_i(y) - \sum_{j=1}^m (f_j(y) + g_j(\bar{y})) + \sum_{j=1}^m f_j(y) \\ &= 2 \sum_{i=1}^m f_i(y) + \sum_{i=m+1}^M f_i - \sum_{j=1}^m (f_j(y) + g_j(\bar{y})) \\ &= (M + m) f_{\tilde{i}}(y) + 2 \sum_{i=1}^m (f_i(y) - f_{\tilde{i}}(y)) + \\ &\quad + \sum_{i=m+1}^M (f_i(y) - f_{\tilde{i}}(y)) - \sum_{j=1}^m (f_j(y) + g_j(\bar{y}))\end{aligned}$$

for each $\tilde{i} \in \{1, \dots, M\}$, and for any $j \in \{1, \dots, m\}$

$$g_j(\bar{y}) = (f_1(y) + g_j(\bar{y})) - f_1(y).$$

The theorem thus follows by (8.1.5) and (10.1.4), (10.1.5). \square

10.2 Theorem

The following theorem will be instrumental in showing the necessary decay rate Theorem 11.2 for the excess; in particular, β as given below will be

used to find the necessary angle ϑ as in Theorem 11.2. As such, we give a slightly more thorough conclusion, in stating specifically that $Df(0) = (0, \beta) = Dg(0)$, than as in the statement of Theorem 8.2 of [14], to which Theorem 10.2 is analogous.

Theorem. *Suppose $m \geq 1$. If $\{T_k\}_{k \in \mathbb{N}}$ is a blowup sequence in \mathcal{T} with associated blowups f_i, g_j , then there exists two functions $f \in C^2(\mathbf{V} \cup \mathbf{L})$ and $g \in C^2(\mathbf{W} \cup \mathbf{L})$ and $\beta \in \mathbb{R}$ such that*

$$\begin{aligned} f|_{\mathbf{V}} &= f_1 = f_2 = \dots = f_M, & g|_{\mathbf{W}} &= g_1 = \dots = g_m \\ f|_{\mathbf{L}} &= 0 = g|_{\mathbf{L}}, & \text{and } Df(0) &= (0, \beta) = Dg(0). \end{aligned}$$

Proof. The proof is precisely the same as in section 8.2 of [14], with only minor notational changes.

By Lemma 10.1, the harmonic functions $f_i^{(\rho)}, g_j^{(\rho)}$ defined for $i \in \{1, \dots, M\}$, $j \in \{1, \dots, m\}$, and $\rho \in (0, 1/4]$ by

$$\begin{aligned} f_i^{(\rho)}(y) &= f_i(\rho y)/\rho \quad \text{for } y \in \mathbf{V}, \\ g_j^{(\rho)}(y) &= g_j(\rho y)/\rho \quad \text{for } y \in \mathbf{W}, \end{aligned}$$

are uniformly bounded. By Theorem 2.11 of [12], there is a decreasing sequence $\{\rho_l\}_{l \in \mathbb{N}} \subset (0, 1/4]$ so that $\rho_l \rightarrow 0$ as $l \rightarrow \infty$ and bounded harmonic functions f_1^*, \dots, f_M^* on \mathbf{V} and g_1^*, \dots, g_m^* on \mathbf{W} such that for all $i \in \{1, \dots, M\}$ and $j \in \{1, \dots, m\}$

$$\begin{aligned} f_i^{(\rho_l)} &\rightarrow f_i^* \quad \text{and} \quad Df_i^{(\rho_l)} \rightarrow Df_i^* \quad \text{pointwise on } \mathbf{V} \text{ as } l \rightarrow \infty, \\ g_j^{(\rho_l)} &\rightarrow g_j^* \quad \text{and} \quad Dg_j^{(\rho_l)} \rightarrow Dg_j^* \quad \text{pointwise on } \mathbf{W} \text{ as } l \rightarrow \infty. \end{aligned}$$

Noting that

$$\begin{aligned} &\sum_{i=1}^M \int_{\mathbf{V}} \left(\frac{\partial}{\partial r} \left(\frac{f_i(y)}{|y|} \right) \right)^2 |y|^{2-n} d\mathcal{H}^n(y) \\ &\quad + \sum_{j=1}^m \int_{\mathbf{W}} \left(\frac{\partial}{\partial r} \left(\frac{g_j(y)}{|y|} \right) \right)^2 |y|^{2-n} d\mathcal{H}^n(y) < \infty \end{aligned}$$

by (7.4.3) and (8.1.2), we use Fatou's lemma to compute for each

$i \in \{1, \dots, M\}$

$$\begin{aligned} & \int_{\mathbf{V}} \left(\frac{\partial}{\partial r} \left(\frac{f_i^*(y)}{|y|} \right) \right)^2 |y|^{2-n} d\mathcal{H}^n(y) \\ & \leq \liminf_{l \rightarrow \infty} \int_{\mathbf{V}} \left(\frac{\partial}{\partial r} \left(\frac{f_i^{(\rho_l)}(y)}{|y|} \right) \right)^2 |y|^{2-n} d\mathcal{H}^n(y) \\ & \leq \liminf_{l \rightarrow \infty} \int_{\mathbf{V} \cap B_{\rho_l}^n} \left(\frac{\partial}{\partial r} \left(\frac{f_i(y)}{|y|} \right) \right)^2 |y|^{2-n} d\mathcal{H}^n(y) = 0, \end{aligned}$$

hence $\frac{\partial}{\partial r} \left(\frac{f_i^*(y)}{|y|} \right) = 0$ for all $y \in \mathbf{V}$. Similarly, $\frac{\partial}{\partial r} \left(\frac{g_j^*(y)}{|y|} \right) = 0$ for all $y \in \mathbf{W}$. As in the proof of Theorem 9.3, it follows from Remark 9.2 that each f_i^* (respectively, g_j^*) is the restriction to \mathbf{V} (respectively, \mathbf{W}) of some multiple of the linear function $\mathbf{q}_0(y) = y_n$. In the next paragraph, we shall verify that all of these multiples coincide.

Define for each $l \in \mathbb{N}$

$$S_k^l = (\boldsymbol{\eta}_{\rho_l \#} T_k) \llcorner B_3,$$

and choose, by Remark 4.3 and (8.1.1), (8.1.2), an integer $N_l \geq l$ so that $S_k^l \in \mathcal{T}$ whenever $k \geq N_l$. Noting that, for each $i \in \{1, \dots, M\}$, $j \in \{1, \dots, m\}$, and $l \in \mathbb{N}$

$$\begin{aligned} \varepsilon_k^{-1} v_i^{S_k^l} &\rightarrow f_i^{(\rho_l)} && \text{uniformly on compact subsets of } \mathbf{V}, \\ \varepsilon_k^{-1} w_j^{S_k^l} &\rightarrow g_j^{(\rho_l)} && \text{uniformly on compact subsets of } \mathbf{W} \end{aligned}$$

as $k \rightarrow \infty$, we may find integers $k_l \geq N_l$ so that

$$\begin{aligned} (10.2.1) \quad & \max \left\{ \sup_{\mathbf{V}_{\frac{1}{2}}} |f_1^{(\rho_l)}|, \sup_{\mathbf{V}_{\frac{1}{2}}} |f_M^{(\rho_l)}|, \sup_{\mathbf{W}_{\frac{1}{2}}} |g_1^{(\rho_l)}|, \sup_{\mathbf{W}_{\frac{1}{2}}} |g_m^{(\rho_l)}| \right\} \\ & \leq \sup_{C_{\frac{1}{2}} \cap \text{spt } S_{k_l}^l} \varepsilon_{k_l}^{-1} |\mathbf{q}_1| + \frac{1}{l}, \end{aligned}$$

and, by applying Lemma 8.3 to $\{T_k\}_{k \in \mathbb{N}}$ with $y = 0$ and $\sigma = 3\rho_l$, so that

$$\begin{aligned} (10.2.2) \quad & \sup_{C_3 \cap \text{spt } S_{k_l}^l} \varepsilon_{k_l}^{-1} |\mathbf{q}_1| \\ & \leq 3 \max \left\{ \sup_{\mathbf{V}} |f_1^{(\frac{\rho_l}{3})}|, \sup_{\mathbf{V}} |f_M^{(\frac{\rho_l}{3})}|, \sup_{\mathbf{W}} |g_1^{(\frac{\rho_l}{3})}|, \sup_{\mathbf{W}} |g_m^{(\frac{\rho_l}{3})}| \right\} + \frac{1}{l}, \end{aligned}$$

and so that, with $S_l^* = S_{k_l}^l$,

$$(10.2.3) \quad \varepsilon_{k_l}^{-1} v_i^{S_l^*} \rightarrow f_i^* \quad \text{and} \quad \varepsilon_{k_l}^{-1} w_j^{S_l^*} \rightarrow g_j^* \quad \text{as } l \rightarrow \infty;$$

these three assumptions are as in 8.2(1)(2)(3) of [14]. In case not all functions f_i^*, g_j^* are identically zero, using (6.1.1), (6.1.2) (with T, σ replaced by $S_l^*, 1/2$) we see that (10.2.1), (10.2.2) along with (4.3.2) and (8.1.2) imply

$$0 < \liminf_{l \rightarrow \infty} \varepsilon_{k_l}^{-1} \mathbf{E}_C(S_l^*, 1)^{\frac{1}{2}} \leq \limsup_{l \rightarrow \infty} \varepsilon_{k_l}^{-1} \mathbf{E}_C(S_l^*, 1)^{\frac{1}{2}} < \infty,$$

and we may use Definition 8.1 and (10.2.2) to obtain a positive number λ and a blowup sequence $\{S_l^*\}_{l \in \mathbb{N}}$ whose associated harmonic blowups are $\lambda f_i^*, \lambda g_j^*$. It follows from Lemma 8.2 and Lemma 8.4 (as in the proof of Theorem 9.3) that, in any case

$$f_1^* = f_2^* = \dots = f_M^* = \beta \mathbf{q}_0|_{\mathbf{V}} \quad g_1^* = \dots = g_m^* = \beta \mathbf{q}_0|_{\mathbf{W}}$$

for some $\beta \in \mathbb{R}$.

This now implies that the nonnegative harmonic functions $f_M - f_1$ and $g_m - g_1$ satisfy the conditions

$$\liminf_{t \searrow 0} \frac{f_M(0, t) - f_1(0, t)}{t} = 0 = \liminf_{t \searrow 0} \frac{g_m(0, -t) - g_1(0, -t)}{t}.$$

By the Hopf boundary point lemma (see Lemma 3.4 of [12]), we conclude

$$f_M - f_1 \equiv 0 \text{ on } \mathbf{V} \text{ and } g_m - g_1 \equiv 0 \text{ on } \mathbf{W};$$

hence,

$$f_1 = f_2 = \dots = f_M \text{ on } \mathbf{V} \text{ and } g_1 = \dots = g_m \text{ on } \mathbf{W}.$$

By Lemma 8.2 these functions all have zero trace on \mathbf{L} . Thus, there exist by the weak version of the Schwarz reflection principle functions $f \in C^2(\mathbf{V} \cup \mathbf{L})$ and $g \in C^2(\mathbf{W} \cup \mathbf{L})$ such that

$$f|_{\mathbf{V}} = f_1 = f_2 = \dots = f_M, \quad g|_{\mathbf{W}} = g_1 = \dots = g_m, \quad f|_{\mathbf{L}} = 0 = g|_{\mathbf{L}}.$$

Moreover, $Df(0) = (0, \beta) = Dg(0)$. □

10.3 Remark

Suppose $m \geq 1$. By the Schwarz reflection principle, well-known L^2 a priori estimates for harmonic functions (see 5.2.5 of [10]), together with (8.1.5) taking $\rho = 1/2$, we conclude there are positive constants c_{42}, c_{43} depending on n, M, m so that for any harmonic blowups f, g as in Theorem 10.2

$$(10.3.1) \quad \begin{aligned} |\beta| &= |Df(0)| = |Dg(0)| \\ &\leq c_{42} \min \left\{ \left(\int_{\mathbf{V} \cap B_{\frac{1}{2}}^n} |f|^2 d\mathcal{H}^n \right)^{\frac{1}{2}}, \left(\int_{\mathbf{W} \cap B_{\frac{1}{2}}^n} |g|^2 d\mathcal{H}^n \right)^{\frac{1}{2}} \right\} \leq c_{43}, \end{aligned}$$

$$(10.3.2) \quad \begin{aligned} &|f(y) - y \cdot Df(0)| \\ &\leq c_{42} \left(\int_{\mathbf{V} \cap B_{\frac{1}{2}}^n} |f|^2 d\mathcal{H}^n \right)^{\frac{1}{2}} |y|^2 \leq c_{43} |y|^2 \text{ for } y \in \mathbf{V} \cap B_{\frac{1}{4}}^n, \end{aligned}$$

$$(10.3.3) \quad \begin{aligned} &|g(y) - y \cdot Dg(0)| \\ &\leq c_{42} \left(\int_{\mathbf{W} \cap B_{\frac{1}{2}}^n} |g|^2 d\mathcal{H}^n \right)^{\frac{1}{2}} |y|^2 \leq c_{43} |y|^2 \text{ for } y \in \mathbf{W} \cap B_{\frac{1}{4}}^n; \end{aligned}$$

these are as in 8.3(1)(2)(3) of [14] (note that section 8.3 of [14] is mislabeled in [14] as section 8.2).

11 Excess growth estimate

Theorem 11.2 is the central cylindrical excess decay lemma we need to prove Corollary 11.3, which together with the Hopf-type boundary point lemma from section 12.1 directly implies our main result Theorem 2. All of the results and proofs hold from section 9 of [14] without change. We introduce c_{44}, c_{45}, c_{46} now depending on n, M, m , and we assume $m \geq 1$.

11.1 Theorem

As noted before, proving a result such as Theorem 11.2 is standard in proving regularity results for area-minimizing currents. Moreover, the proof of Theorem 11.2 is a standard iteration argument. For this, we first require Theorem 11.1. The proof of the following is exactly the same as in section 9.1 of [14].

Theorem. *Suppose $m \geq 1$. There is a constant $c_{44} \geq 1$ depending on n, M, m so that corresponding to each $T \in \mathcal{T}$ with $\max\{\mathbf{E}_C(T, 1), c_{44}\kappa_T\} \leq c_{44}^{-1}$, there exists $\theta \in \mathbb{R}$ for which*

$$|\theta| \leq c_{43} \mathbf{E}_C(T, 1)^{\frac{1}{2}}$$

$$\mathbf{E}_C(\mathbf{rot}_{\theta\#} T, \tau) \leq \tau \max\{\mathbf{E}_C(T, 1), c_{44}\kappa_T\},$$

where $\tau = (c_{16}(1 + c_{43}))^{-2/\alpha}$, with c_{16} as in (6.2.4) and c_{43} as in Remark 10.3; thus τ depends only on n, M, m .

Proof. If the theorem were false, then there would exist for each $k \in \mathbb{N}$ a current $T_k \in \mathcal{T}$ so that, with $\varepsilon_k = \mathbf{E}_C(T_k, 1)^{\frac{1}{2}}$ and $\kappa_k = \kappa_{T_k}$,

$$(11.1.1) \quad \max\{\varepsilon_k^2, k \kappa_k\} \leq k^{-1}$$

$$(11.1.2) \quad \mathbf{E}_C(\mathbf{rot}_{\theta\#} T_k, \tau) > \tau \max\{\varepsilon_k^2, k \kappa_k\} \text{ whenever } |\theta| \leq c_{43}\varepsilon_k;$$

these are as in 9.1(1)(2) of [14].

We assume as in 9.1(3) of [14] that

$$(11.1.3) \quad k \kappa_k < \tau^{-n-1} \varepsilon_k^2 \text{ for all } k \in \mathbb{N},$$

otherwise (11.1.2) is, by (3.2.1), contradicted with $\theta = 0$.

Replacing $\{T_k\}_{k \in \mathbb{N}}$ by a subsequence, we may assume by Definition 8.1 that $\{T_k\}_{k \in \mathbb{N}}$ is a blowup sequence with associated harmonic blowups f, g as in Theorem 10.2. By Theorem 10.2 and (10.3.1),

$$(11.1.4) \quad Df(0) = (0, \beta) = Dg(0) \text{ for some } \beta \in [-c_{43}, c_{43}];$$

this is as in 9.1(4) of [14]. Letting $\theta_k = \arctan(\beta \varepsilon_k)$ so that

$$(11.1.5) \quad |\theta_k| \leq |\beta| \varepsilon_k \leq c_{43} \varepsilon_k,$$

as in 9.1(5) of [14], we infer from (11.1.1) and (6.2.4) that

$$(11.1.6) \quad (\boldsymbol{\eta}_{\tau\#} \mathbf{rot}_{\theta_k\#} T_k) \llcorner B_3 \in \mathcal{T} \quad \text{and} \quad \kappa_{(\boldsymbol{\eta}_{\tau\#} \mathbf{rot}_{\theta_k\#} T_k) \llcorner B_3} \leq \tau^\alpha \kappa_k$$

for all $k \in \mathbb{N}$ sufficiently large; this is as in 9.1(6) of [14].

Noting that $f|_{\mathbf{L}} = 0 = g|_{\mathbf{L}}$ and that the functions $\varepsilon_k^{-1} v_i^{T_k}, \varepsilon_k^{-1} w_j^{T_k}$ converge uniformly as $k \rightarrow \infty$ on $\mathbf{V}_\sigma, \mathbf{W}_\sigma$ respectively, for each $\sigma \in (0, 1)$ we conclude from Lemma 8.3 that

$$\begin{aligned} \lim_{k \rightarrow \infty} \sup_{C_{\frac{1}{2}} \cap \mathbf{p}^{-1}(\mathbf{V}) \cap \text{spt } T_k} |\varepsilon_k^{-1} \mathbf{q}_1 - f \circ \mathbf{p}| &= 0, \\ \lim_{k \rightarrow \infty} \sup_{C_{\frac{1}{2}} \cap \mathbf{p}^{-1}(\mathbf{W}) \cap \text{spt } T_k} |\varepsilon_k^{-1} \mathbf{q}_1 - g \circ \mathbf{p}| &= 0. \end{aligned}$$

From these equations, (11.1.4), and (10.3.2),(10.3.3) we deduce that

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \sup_{C_2 \cap \text{spt}(\boldsymbol{\eta}_{\tau\#} \mathbf{rot}_{\theta_k\#} T_k)} \varepsilon_k^{-1} |\mathbf{q}_1| \\ & \leq \tau^{-1} \limsup_{k \rightarrow \infty} \sup_{C_{3\tau} \cap \text{spt } T_k} |\varepsilon_k^{-1} \mathbf{q}_1 - \beta \mathbf{q}_0| \\ & \leq \tau^{-1} \limsup_{k \rightarrow \infty} \left(\sup_{C_{3\tau} \cap \mathbf{p}^{-1}(\mathbf{V}) \cap \text{spt } T_k} |f \circ \mathbf{p} - \beta \mathbf{q}_0| \right. \\ & \quad \left. + \sup_{C_{3\tau} \cap \mathbf{p}^{-1}(\mathbf{W}) \cap \text{spt } T_k} |g \circ \mathbf{p} - \beta \mathbf{q}_0| \right) \\ & \leq 18c_{43}\tau. \end{aligned}$$

Using this estimate, (11.1.1),(11.1.3),(11.1.6), and (6.1.1) (with $\sigma \nearrow 1$), we conclude that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \varepsilon_k^{-2} \mathbf{E}_C(\mathbf{rot}_{\theta_k\#} T_k, \tau) &= \limsup_{k \rightarrow \infty} \varepsilon_k^{-2} \mathbf{E}_C(\boldsymbol{\eta}_{\tau\#} \mathbf{rot}_{\theta_k\#} T_k, 1) \\ &\leq (18)^2 c_{12} c_{13} c_{43}^2 \tau^2 < \theta, \end{aligned}$$

which, along with (11.1.5), contradicts (11.1.2). \square

11.2 Theorem

Finally, we attend to the key excess decay estimate.

Theorem. Suppose $m \geq 1$, and let c_{43}, c_{44} , and τ be as in Remark 10.3 and Theorem 11.1. For any $T \in \mathcal{T}$ with $\max\{\mathbf{E}_C(T, 1), c_{44}\kappa_T\} \leq c_{44}^{-1}\tau^\alpha$, there exists a $\vartheta \in [-2c_{43}c_{44}^{-1/2}\tau^{\alpha/2}, 2c_{43}c_{44}^{-1/2}\tau^{\alpha/2}]$ such that

$$\mathbf{E}_C(\mathbf{rot}_{\vartheta\#}T, \rho) \leq c_{44}^{-1}\vartheta^{-(n+2)\alpha}\rho^\alpha \text{ for } \rho \in (0, \tau/4).$$

Proof. The proof is a standard inductive argument using Theorem 11.1, and we closely follow the proof in section 9.2 of [14].

By Theorem 11.1 and (6.2.4) we may inductively select real numbers $\{\theta_k\}_{k \in \mathbb{N}}$ and rectifiable currents $\{T_k\}_{k \in \mathbb{N}}$ so that, after setting $T_0 = T$, we have for each $k \in \mathbb{N}$

$$\begin{aligned} |\theta_k| &\leq c_{43}\mathbf{E}_C(T_{k-1}, 1)^{\frac{1}{2}}, \\ T_k &= (\eta_{\tau\#}\mathbf{rot}_{\theta_k\#}T_{k-1}) \llcorner B_3 \in \mathcal{T}, \text{ and} \\ \max\{\mathbf{E}_C(T_k, 1), c_{44}\kappa_{T_k}\} &\leq \tau^\alpha \max\{\mathbf{E}_C(T_{k-1}, 1), c_{44}\kappa_{T_{k-1}}\} \leq c_{44}^{-1}\tau^{(j+1)\alpha}. \end{aligned}$$

In inductively checking that T_k belongs to \mathcal{T} , we use (6.2.4) and the definition of τ in Theorem 11.1 to estimate

$$(11.2.1) \quad |\theta_k| \leq c_{43}c_{44}^{-1/2}\tau^{k\alpha/2},$$

$$(11.2.2) \quad \mathbf{E}_C(T_{k-1}, 1) + \kappa_{T_{k-1}} \leq 2 \max\{\mathbf{E}_C(T_{k-1}, 1), \kappa_{T_{k-1}}\} \leq 2c_{44}^{-1}\tau^{j\alpha};$$

these are as in 9.2(1)(2) of [14]. Letting $\vartheta_k = \sum_{l=1}^k \theta_l$ and $\vartheta = \sum_{l=1}^\infty \theta_l$, we notice that by (11.2.1)

$$|\vartheta| \leq 2c_{43}c_{44}^{-1/2}\tau^{\alpha/2}.$$

For any $\rho \in (0, \tau/4)$, we may choose $k \in \mathbb{N}$ so that $\rho \in [\frac{\tau^{k+1}}{4}, \frac{\tau^j}{4})$, and apply (11.2.1), (11.2.2) (with k replaced by $k+1$), (6.2.3), (6.2.4) (with T, θ replaced by $T_k, \vartheta - \vartheta_k$) and (3.2.1) to estimate

$$\begin{aligned} \mathbf{E}_C(\mathbf{rot}_{\vartheta\#}T, \rho) &\leq \tau^{-n\alpha}\mathbf{E}_C(\mathbf{rot}_{\vartheta\#}T, \tau^k/4) = 4^n\tau^{-n\alpha}\mathbf{E}_C(\eta_{\frac{1}{4}\#}\mathbf{rot}_{\vartheta\#}T, \tau^j) \\ &= 4^n\tau^{-n\alpha}\mathbf{E}_C(\eta_{\frac{1}{4}\#}\mathbf{rot}_{(\vartheta-\vartheta_k)\#}T_k, 1) \\ &\leq \tau^{-(n+1)\alpha} \left(\left(\sum_{l=k+1}^\infty |\theta_l|^2 \right) + \mathbf{E}_C(T_k, 1) + \kappa_{T_k} \right) \\ &\leq c_{44}^{-1}\tau^{-(n+2)\alpha}\rho^\alpha. \end{aligned}$$

□

11.3 Corollary

The following is essentially Theorem 2, prior to an application of the Hopf-type boundary point Lemma 12.1.

Corollary. *Suppose $m \geq 1$. If T and ϑ are as in Theorem 11.2, if $a = \alpha/(2n+6)$, and if*

$$\begin{aligned}\tilde{V} &= \{y \in B_\delta^n : y_n > |y|^{1+a}\}, \\ \tilde{W} &= \{y \in B_\delta^n : y_n < -|y|^{1+a}\},\end{aligned}$$

where $\delta = ((4c_{24})^{-2n-3}\tau^{(n+2)\alpha})^{\frac{2}{\alpha}}$ and c_{24}, τ are as in Theorem 7.4 and Theorem 11.1, then

$$\begin{aligned}\mathbf{p}^{-1}(\tilde{V}) \cap \text{spt } \mathbf{rot}_{\vartheta\#} T &= \bigcup_{i=1}^M \text{graph}_{\tilde{V}} \tilde{v}_i, \\ \mathbf{p}^{-1}(\tilde{W}) \cap \text{spt } \mathbf{rot}_{\vartheta\#} T &= \bigcup_{j=1}^m \text{graph}_{\tilde{W}} \tilde{w}_j\end{aligned}$$

for some $\tilde{v}_i \in C^{1,a}(\text{Clos } \tilde{V})$, $\tilde{w}_j \in C^{1,a}(\text{Clos } \tilde{W})$ such that $\tilde{v}_i|_{\tilde{V}}$, $\tilde{w}_j|_{\tilde{W}}$ satisfy the minimal surface equation, $D\tilde{v}_i(0) = 0 = D\tilde{w}_j(0)$, and

$$\tilde{v}_1 \leq \tilde{v}_2 \leq \dots \leq \tilde{v}_M, \quad \tilde{w}_1 \leq \dots \leq \tilde{w}_m.$$

Proof. For each $\rho \in (0, \delta)$ we note that, as in the proof of Theorem 11.2, the current $S_\rho = (\boldsymbol{\eta}_{\rho\#} \mathbf{rot}_{\vartheta\#} T) \llcorner B_3$ belongs to \mathcal{T} and that, by Theorem 7.4, Theorem 11.2, and (4.3.2),

$$\begin{aligned}(11.3.1) \quad \sigma_{S_\rho} &\leq c_{24}(\mathbf{E}_C(S_\rho, 1) + \kappa_{S_\rho})^{\frac{1}{2n+3}} \leq c_{24}((\tau^{-(n+2)\alpha} + \tau^\alpha)\rho^\alpha)^{\frac{1}{2n+3}} \\ &\leq c_{24}(2\tau^{-(n+2)\alpha}\rho^{\alpha/2})^{\frac{1}{2n+3}}\rho^a \leq \rho^a/4;\end{aligned}$$

this is as in 9.3(1) of [14]. Applying (7.4.1) (with $T = S_\rho$ and $l = 1, 2$), we infer that

$$(11.3.2) \quad \sup_{\mathbf{V}_{\rho^a/4}} |Dv_i^{S_\rho}| \leq c_{25}(2\theta^{-(n+2)\alpha}\rho^\alpha)^{1/2}(\rho^a/4)^{-1} \leq c_{45}\rho^a,$$

$$(11.3.3) \quad \sup_{\mathbf{V}_{\rho^{\beta/4}}} |D^2v_i^{S_\rho}| \leq c_{25}(2\theta^{-(n+2)\alpha}\rho^\alpha)^{1/2}(\rho^a/4)^{-2} \leq c_{46}\rho^a,$$

for each $i = 1, \dots, M$, with c_{45}, c_{46} depending on n, M, m ; these are in 9.3(2)(3) of [14].

Next, observe that if we take $y \in \tilde{V}$ and if we take $\rho \in (0, 2|y|)$, then (by definition of \tilde{V})

$$(11.3.4) \quad y_n/\rho > \rho^a/4 \quad (\sigma_{S_\rho} \text{ by (11.3.1)});$$

this is as in 9.3(4) of [14]. Hence, using (11.3.1), it follows that

$$\mathbf{p}^{-1}(\tilde{V}) \cap \text{spt } \mathbf{rot}_{\vartheta^\#} T = \bigcup_{i=1}^M \text{graph}_{\tilde{V}} \tilde{v}_i$$

for some functions $\tilde{v}_i : \tilde{V} \rightarrow \mathbb{R}$ which are well-defined by the condition

$$(11.3.5) \quad \tilde{v}_i(y) = \rho v_i^{S_\rho}(y/\rho) \text{ whenever } y \in \tilde{V} \text{ and } \frac{3}{2}|y| \leq \rho \leq 2|y|;$$

this is as in 9.3(5) of [14]. By Theorem (7.4), $\tilde{v}_i \leq \tilde{v}_2 \leq \dots \leq \tilde{v}_M$ and each \tilde{v}_i satisfies the minimal surface equation on \tilde{V} . Moreover, we may use (11.3.4), (11.3.5), (11.3.2), (11.3.3) (with $\rho = 2|y|$) to estimate

$$(11.3.6) \quad |D\tilde{v}_i(y)| \leq 2c_{45}|y|^a,$$

$$(11.3.7) \quad |D^2\tilde{v}_i(y)| \leq 2c_{46}|y|^a(2|y|)^{-1} = c_{46}|y|^{a-1}$$

for any $y \in \tilde{V}$; these are as in 9.3(6)(7) of [14].

To obtain the desired Hölder, we suppose $y, \tilde{y} \in \tilde{V}$ and consider the two possibilities:

Case 1. $\max\{|y|, |\tilde{y}|\} \leq 2|y - \tilde{y}|$. Here we infer from (11.3.6) that

$$(11.3.8) \quad \begin{aligned} |D\tilde{v}_i(y) - D\tilde{v}_i(\tilde{y})| &\leq |D\tilde{v}_i(y)| + |D\tilde{v}_i(\tilde{y})| \\ &\leq 2c_{45}(|y|^a + |\tilde{y}|^a) \\ &\leq 4c_{45}|y - \tilde{y}|^a; \end{aligned}$$

this is as in 9.3(8) of [14].

Case 2. $\max\{|y|, |\tilde{y}|\} > 2|y - \tilde{y}|$. Here

$$|y + t(\tilde{y} - y)| > |y - \tilde{y}| \text{ whenever } t \in [0, 1],$$

and so, by (11.3.7),

$$(11.3.9) \quad \begin{aligned} |D\tilde{v}_i(y) - D\tilde{v}_i(\tilde{y})| &\leq |y - \tilde{y}| \int_0^1 |D^2\tilde{v}_i(y + t(\tilde{y} - y))| dt \\ &\leq c_{46}|y - \tilde{y}|^a; \end{aligned}$$

this is 9.3(9) of [14].

By (11.3.8), (11.3.9), and (11.3.6) each function \tilde{v}_i extends uniquely to a member of $C^{1,a}(\text{Clos } \tilde{V})$ with $D\tilde{v}_i(0) = 0$.

The argument to show the existence of $\tilde{w}_i \in C^{1,a}(\text{Clos } \tilde{W})$ is similar. \square

12 A Hopf-type boundary point lemma

Here we give a short proof of a Hopf-type boundary point lemma for divergence-form elliptic equations, which was first established by Finn and Gilbarg; see Lemma 7 of [11]. This is as in section 10 of [14].

We use in section 12.1 positive constants c_{47}, c_{48}, c_{49} from applying Theorem 5.5.5'(b) of [17]; while these constants do not depend only on n, M, m , we will only use these constants in section 12.1.

12.1 Lemma

Lemma. *If $a \in (0, 1)$, Ω is a connected $C^{1,a}$ domain in \mathbb{R}^n , $a_{kl} \in C^{0,a}(\text{Clos } \Omega)$ for $k, l \in \{1, \dots, n\}$,*

$$\sum_{k,l=1}^n a_{kl}(y) \xi_k \xi_l \geq |\xi|^2 \text{ for } y \in \Omega \text{ and } \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n,$$

$u \in C^{1,a}(\text{Clos } \Omega, [0, \infty))$ is a weak solution of the equation

$$\sum_{k,l=1}^n D_k(a_{kl} D_l u) = 0$$

on Ω , $0 \in \partial\Omega$, and $u(0) = 0$, then

$$\text{either } u \equiv 0 \text{ or } \mathbf{n} \cdot Du(0) > 0$$

where \mathbf{n} is the interior unit normal to Ω at 0.

Proof. We follow the proof of Lemma 10.1 of [14], with only changes in notation.

By making a nonsingular $C^{1,a}$ transformation of coordinates near 0 we may assume $\Omega \cap B_1^n = \mathbf{V}$; hence $\mathbf{n} \cdot Du(0) = D_n u(0)$. Let

$$A = B_{\frac{1}{2}}^n(e_n/2) \setminus \bar{B}_{\frac{1}{4}}^n(e_n/2)$$

and let ϕ be the analytic function on $\text{Clos } A$ for which

$$\begin{aligned} \phi|_{\partial B_{\frac{1}{2}}^n(e_n/2)} &\equiv 0 & \phi|_{\partial \bar{B}_{\frac{1}{4}}^n(e_n/2)} &\equiv 1 \\ \sum_{k,l=1}^n a_{kl}(0) D_k D_l \phi &\equiv 0. \end{aligned}$$

By the classical Hopf argument (see, for example, Theorem 3.6 of [12]) for nondivergence form equations,

$$(12.1.1) \quad D_n \phi(0) > 0;$$

this is as in 10.1(1) of [14].

Next, for each $\epsilon \in (0, 1)$, let

$$u^\epsilon(y) = u(\epsilon y) \text{ and } a_{kl}^\epsilon(y) = a_{kl}(\epsilon y) \text{ for } y \in \mathbf{V} \text{ and } \{k, l\} \subseteq \{1, \dots, n\},$$

and note that $a_{kl}^\epsilon(0) = a_{kl}(0)$ and that u^ϵ is a weak solution of the equation

$$(12.1.2) \quad \sum_{k,l=1}^n D_k(a_{kl}^\epsilon D_l w) = 0$$

on \mathbf{V} ; this is as in 10.1(2) of [14]. By section 23 of [28] and Theorem 5.5.4' of [17], we may choose $\phi_\epsilon \in C^{1,a}(\text{Clos } A)$ a weak solution of (12.1.2) on A so that $\phi_\epsilon|_{\partial A} = \phi|_{\partial A}$. Using the functions

$$\zeta_k^\epsilon(y) = - \sum_{l=1}^n (a_{kl}^\epsilon(y) - a_{kl}^\epsilon(0)) D_l \phi(y) \text{ for } y \in \text{Clos } A \text{ and } k \in \{1, \dots, n\},$$

we observe that $\psi_\epsilon = \phi_\epsilon - \phi \in C^{1,a}(\text{Clos } A)$ is a weak solution of the equation

$$(12.1.3) \quad \sum_{k,l=1}^n D_k(a_{kl}^\epsilon D_l w) = \sum_{k=1}^n D_k \zeta_k^\epsilon$$

on A with $\psi_\epsilon|_{\partial A} = 0$; this is as in 10.1(3) of [14]. We infer from Theorem 5.5.5'(b) of [17]

$$(12.1.4) \quad \|\psi_\epsilon\|_{C^{1,a}(\text{Clos } A)} \leq c_{47} \left(\sum_{k=1}^n \|\zeta_k^\epsilon\|_{C^{0,a}(\text{Clos } A)} + \int_A |\psi_\epsilon| d\mathcal{H}^n \right),$$

where c_{47} is a constant independent of ϵ ; this is as in 10.4 of [14]. Moreover, using the coercivity of the operator on the left of (12.1.3) as in section 23 of [28], we readily verify that

$$(12.1.5) \quad \int_A |\psi_\epsilon|^2 d\mathcal{H}^n \leq c_{48} \sum_{k=1}^n \int_A |\zeta_k^\epsilon|^2 d\mathcal{H}^n \leq c_{49} \sum_{k=1}^n \|\zeta_k^\epsilon\|_{C^{0,a}(\text{Clos } A)}^2$$

with c_{48}, c_{49} independent of ϵ ; this is as in 10.1(5) of [14]. Combining (12.1.4), (12.1.5), and Schwartz's inequality, we conclude that

$$|D_n \phi_\epsilon(0) - D_n \phi(0)| \leq \|\psi_\epsilon\|_{C^{1,a}(\text{Clos } A)} \leq c_{47}(1 + \sqrt{c_{49}}) \sum_{l=1}^n \|\zeta_l^\epsilon\|_{C^{0,a}(\text{Clos } A)} \rightarrow 0$$

as $\epsilon \rightarrow 0$. Thus we may, by (12.1.1), fix $\epsilon > 0$ sufficiently small so that

$$(12.1.6) \quad D_n \phi_\epsilon(0) > \frac{1}{2} D_n \phi(0) > 0;$$

this is as in 10.1(6) of [14].

In case $u|_\Omega$ is strictly positive we can choose $\lambda \in (0, 1)$ so that $(u^\epsilon - \lambda \phi_\epsilon)|_{\partial A} \geq 0$. Since $u^\epsilon - \lambda \phi_\epsilon$ is a solution of (12.1.2), we may infer from the weak maximum principle (see, for example, Theorem 3.6 of [12]) that $D_n(u^\epsilon - \lambda \phi_\epsilon)(0) \geq 0$; hence, by (12.1.6),

$$D_n u(0) = \epsilon D_n u^\epsilon(0) \geq \epsilon \lambda D_n \phi_\epsilon(0) > 0.$$

To complete the proof we will verify that if $u|_\Omega$ is not identically zero, then $u|_\Omega$ is strictly positive. Otherwise, there is $\bar{B}_\rho(y) \subset \Omega$ and a point $\tilde{y} \in \partial B_\rho(y)$ so that $u|_{B_\rho(y)} > 0$ and $u(\tilde{y}) = 0$. Since $u \geq 0$ we infer $Du(\tilde{y}) = 0$, contradicting the argument above with Ω replaced by $B_\rho(y)$. \square

13 Concluding Theorem 2

Having established Corollary 11.3, the proof of Theorem 2 follows exactly as in the first part of Theorem 11.1 of [14]. Recall that $m \geq 1$ in Theorem 2.

Proof of Theorem 2: Suppose $T \in \mathcal{R}^n(B_3)$ satisfies $(*)$, $(**)$ from Theorem 2. Choose $r_k \rightarrow 0$ with $r_k < 1$ so that as currents

$$\boldsymbol{\eta}_{r_k\#} T \rightarrow M\mathbb{E}^n \llcorner \{y \in \mathbb{R}^n : y_n > 0\} + m\mathbb{E}^n \llcorner \{y \in \mathbb{R}^n : y_n < 0\}.$$

By 5.4.2 of [10]

$$\lim_{k \rightarrow \infty} \sup_{\bar{B}_{r_k} \cap \text{spt } T} \mathbf{q}_1 / r_k = 0,$$

and so we can choose $k \in \mathbb{N}$ sufficiently large so that

$$\begin{aligned} T_k &= (\boldsymbol{\eta}_{r_k\#} T) \llcorner B_3 \in \mathcal{T} \\ \max\{\mathbf{E}_C(T_k, 1), c_{44}\kappa_{T_k}\} &\leq c_{44}^{-1}. \end{aligned}$$

It follows that Theorem 11.2 holds for T_k specifically with $\vartheta = 0$. By Lemma 12.1 we conclude in applying Theorem 11.2 to T_k that $\tilde{v}_1 = \tilde{v}_2 = \dots = \tilde{v}_M$ and $\tilde{w}_1 = \dots = \tilde{w}_m$. Together with (11.3.7), we now have Theorem 2. \square

A Appendix

In this section we present some calculations based on the homotopy formula 4.1.9 of [10]. The identities presented in this Appendix will be needed in (3.3.5), Lemma 6.1, Lemma 8.2, and Lemma 8.4.

We begin with the following lemma, which follows from the constancy theorem (see Theorem 26.27 of [26]) and induction, which will be used to prove Lemma A.0.5.

Lemma A.0.1. *Let $\rho \in (0, \infty)$ and $\sigma \in (0, 1)$. Suppose $P \in \mathcal{R}_n(\mathbb{R}^n)$ is nonzero with*

$$\begin{aligned} \text{spt } P \cap \{y \in \mathbb{R}^n : |(y_1, \dots, y_{n-1})| < \rho, |y_n| < 1\} \\ \subset \{y \in \mathbb{R}^n : |(y_1, \dots, y_{n-1})| < \rho, |y_n| < \sigma\} \end{aligned} \tag{A.0.2}$$

and

$$\begin{aligned}
& \partial P \llcorner \{y \in \mathbb{R}^n : |(y_1, \dots, y_{n-1})| < \rho, |y_n| < 1\} \\
(A.0.3) \quad & = (-1)^n \sum_{\ell=1}^N m_\ell \Phi_{P, \ell^\#}(\mathbb{E}^{n-1} \llcorner B_\rho^{n-1}) \\
& \quad + (-1)^{n-1} m_0 \mathbb{E}^{n-1} \llcorner B_\rho^{n-1}
\end{aligned}$$

where $N, m_0, m_1, \dots, m_N \in \mathbb{N}$ satisfy $\sum_{\ell=1}^N m_\ell = m_0$, and for each $\ell = 1, \dots, N$ the map $\Phi_{P, \ell} \in C^1(B_\rho^{n-1}; \mathbb{R}^n)$ is given by $\Phi_{P, \ell}(z) = (z, \varphi_{P, \ell}(z))$ where $\varphi_{P, \ell} \in C^1(B_\rho^{n-1})$ with $\sup_{B_\rho^{n-1}} |\varphi_{P, \ell}| < \sigma$.

Then there is a nonempty $K \subseteq \mathbb{N}$ such that for each $k \in K$ there is an open nonempty connected set $O_{P, k} \subset \{y \in \mathbb{R}^n : |(y_1, \dots, y_{n-1})| < \rho, |y_n| < \sigma\}$ and an integer $m_{O_{P, k}} \neq 0$ so that

$$P \llcorner \{y \in \mathbb{R}^n : |(y_1, \dots, y_{n-1})| < \rho, |y_n| < 1\} = \sum_{k \in K} m_{O_{P, k}} \mathbb{E}^n \llcorner O_{P, k}.$$

Moreover, $m_{O_{P, k}} \in [-m_0, 0)$ if $O_{P, k} \cap \{y \in \mathbb{R}^n : y_n > 0\} \neq \emptyset$, while $m_{O_{P, k}} \in (0, m_0]$ if $O_{P, k} \cap \{y \in \mathbb{R}^n : y_n < 0\} \neq \emptyset$.

Proof. We prove this by induction on n . Note first that the constancy theorem (see Theorem 26.27 of [26]) implies

$$(A.0.4) \quad P \llcorner \{y \in \mathbb{R}^n : |(y_1, \dots, y_{n-1})| < \rho, |y_n| < 1\} = \sum_{k \in K} m_{O_{P, k}} \mathbb{E}^n \llcorner O_{P, k}$$

for some nonempty $K \subseteq \mathbb{N}$, where $m_{O_{P, k}} \neq 0$ is an integer and $O_{P, k}$ is a nonempty open connected subset of

$$\{y \in \mathbb{R}^n : |(y_1, \dots, y_{n-1})| < \rho, |y_n| < \sigma\} \setminus \bigcup_{\ell=1}^N \Phi_{P, \ell}(B_\rho^{n-1})$$

for each $k \in K$. We now begin our proof by induction.

n=2: Define $\tilde{\varphi}_{P, 1}, \dots, \tilde{\varphi}_{P, N} \in C((-\rho, \rho))$ so that

$$\tilde{\varphi}_{P, 1} \leq \tilde{\varphi}_{P, 2} \leq \dots \leq \tilde{\varphi}_{P, N-1} \leq \tilde{\varphi}_{P, N}$$

and so that for each $z \in (-\rho, \rho)$ we have

$$\{\tilde{\varphi}_{P,\tilde{\ell}}(z)\}_{\tilde{\ell}=1}^N = \{\varphi_{P,\ell}(z)\}_{\ell=1}^N.$$

Take $O_{P,k}$ as in (A.0.4), and suppose $O_{P,k} \cap \{y \in \mathbb{R}^2 : y_2 > 0\} \neq \emptyset$. The constancy theorem together with (A.0.2),(A.0.3) imply there is an open interval $I_k \subset (-\rho, \rho)$ and an $\tilde{\ell}_k \in \{1, \dots, N\}$ so that

$$O_{P,k} = \{y \in \mathbb{R}^2 : y_1 \in I_k, y_2 \in (\max\{0, \tilde{\varphi}_{P,\tilde{\ell}_k-1}(y_1)\}, \tilde{\varphi}_{P,\tilde{\ell}_k}(y_1))\}.$$

First, suppose $\tilde{\varphi}_{P,\tilde{\ell}_k}(z) = \tilde{\varphi}_{P,N}(z)$ for each $z \in I_k$. It follows we can find $\ell_1, \dots, \ell_{N-\tilde{\ell}_k+1} \in \{1, \dots, N\}$ so that

$$\varphi_{P,\ell_1}(z) = \dots = \varphi_{P,\ell_{N-\tilde{\ell}_k+1}}(z) = \tilde{\varphi}_{P,N}(z)$$

for each $z \in I_k$, and hence

$$\begin{aligned} \partial O_{P,k} \cap \{y \in \mathbb{R}^2 : \max\{0, \tilde{\varphi}_{P,\tilde{\ell}_k-1}(y_1)\} < y_2\} \\ = \Phi_{P,\ell_1}(I_k) = \dots = \Phi_{P,\ell_{N-\tilde{\ell}_k+1}}(I_k). \end{aligned}$$

From this $m_{O_{P,k}} = -(m_{\ell_1} + \dots + m_{\ell_{N-\tilde{\ell}_k+1}})$ follows, and so $m_{O_{P,k}} \in [-m_0, 0)$.

Second, suppose $\tilde{\varphi}_{P,\tilde{\ell}_k}(z) = \tilde{\varphi}_{P,N-1}(z)$ for each $z \in I_k$, but $\tilde{\varphi}_{P,N-1}(z) < \tilde{\varphi}_{P,N}(z)$ for some $z \in I_k$. We can thus find an open interval $\tilde{I}_k \subset I_k$ and an $O_{P,\tilde{k}}$ from (A.0.4) disjoint from $O_{P,k}$ so that

$$\begin{aligned} \partial O_{P,\tilde{k}} \cap \{y \in \mathbb{R}^2 : y_1 \in \tilde{I}_k, y_2 > 0\} \\ = \{(z, \tilde{\varphi}_{P,N-1}(z)) : z \in \tilde{I}_k\} \cup \{(z, \tilde{\varphi}_{P,N}(z)) : z \in \tilde{I}_k\}. \end{aligned}$$

The previous paragraph applied to $O_{P,\tilde{k}}$ implies there are $\ell_1, \dots, \ell_{N-\tilde{\ell}_k+1} \in \{1, \dots, N\}$ so that

$$m_{O_{P,k}} = -(m_{\ell_1} + \dots + m_{\ell_{N-\tilde{\ell}_k+1}}),$$

and from this $m_{O_{P,k}} \in [-m_0, 0)$ follows.

Third, we can argue inductively that $m_{O_{P,k}} \in [-m_0, 0)$ whenever $O_{P,k} \cap \{y \in \mathbb{R}^2 : y_2 > 0\} \neq \emptyset$. By likewise first considering $\tilde{\varphi}_{P,1}$, we can show $m_{O_{P,k}} \in (0, m_0]$ whenever $O_{P,k} \cap \{y \in \mathbb{R}^2 : y_2 < 0\} \neq \emptyset$. This shows the lemma in case $n = 2$.

n > 2. With $\mathbf{p}_1 : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ given by $\mathbf{p}_1(x) = x_1$, take any of the Lebesgue almost-every $t \in (-\rho, \rho)$ such that the slice

$$\langle P, \mathbf{p}_1, t \rangle = \partial(P \llcorner \{x \in \mathbb{R}^{n+1} : x_1 < t\}) - (\partial P) \llcorner \{x \in \mathbb{R}^{n+1} : x_1 < t\}$$

exists, by 4.3.6 of [10]. Then the following two facts also hold for Lebesgue almost-every $t \in (-\rho, \rho)$. First, by (A.0.2),

$$\begin{aligned} \text{spt} \langle P, \mathbf{p}_1, t \rangle \cap \{y \in \mathbb{R}^n : |(y_1, \dots, y_{n-1})| < \rho, |y_n| < 1\} \\ = \text{spt} \langle P, \mathbf{p}_1, t \rangle \cap \{y \in \mathbb{R}^n : y_1 = t, |(y_2, \dots, y_{n-1})| < \rho - t, |y_n| < 1\} \\ \subset \{y \in \mathbb{R}^n : y_1 = t, |(y_2, \dots, y_{n-1})| < \rho - t, |y_n| < \sigma\}. \end{aligned}$$

Second, by (A.0.3) and Lemma 28.5(3) of [26],

$$\begin{aligned} \partial \langle P, \mathbf{p}_1, t \rangle \llcorner \{y \in \mathbb{R}^n : y_1 = t, |(y_2, \dots, y_{n-1})| < \rho - t, |y_n| < 1\} \\ = (-1)^{n-1} \sum_{\ell=1}^N m_\ell \Phi_{P, \ell^\sharp}(\mathbb{E}^{t, n-2} \llcorner \{z \in \mathbb{R}^{n-1} : z_1 = t, |(z_2, \dots, z_{n-1})| < \rho - t\}) \\ + (-1)^{n-2} m_0 \mathbb{E}^{t, n-2} \llcorner \{z \in \mathbb{R}^{n-1} : z_1 = t, |(z_2, \dots, z_{n-1})| < \rho - t\} \end{aligned}$$

where $\mathbb{E}^{t, n-2}$ is the $(n-2)$ -dimensional current in \mathbb{R}^n given by

$$\mathbb{E}^{t, n-2}(\omega) = \int_{\{z \in \mathbb{R}^{n-1} : z_1 = t\}} \langle \omega, e_2 \wedge \dots \wedge e_{n-1} \rangle d\mathcal{H}^{n-2} \text{ for } \omega \in \mathcal{D}^{n-2}(\mathbb{R}^n).$$

These two facts imply by induction that whenever $\langle P, \mathbf{p}_1, t \rangle \neq 0$

$$\begin{aligned} \langle P, \mathbf{p}_1, t \rangle \llcorner \{y \in \mathbb{R}^n : y_1 = t, |(y_2, \dots, y_{n-1})| < \rho - t, |y_n| < 1\} \\ = \sum_{k \in K^t} m_{O_{P,k}^t} \mathbb{E}^{t, n-1} \llcorner O_{P,k}^t \end{aligned}$$

where $\mathbb{E}_t^{t, n-1}$ is the $(n-1)$ -dimensional current in \mathbb{R}^n given by

$$\mathbb{E}_t^{t, n-1}(\omega) = \int_{\{y \in \mathbb{R}^n : y_1 = t\}} \langle \omega, e_2 \wedge \dots \wedge e_n \rangle d\mathcal{H}^{n-1} \text{ for } \omega \in \mathcal{D}^{n-1}(\mathbb{R}^n),$$

and where $K^t \subseteq \mathbb{N}$ is a nonempty set such that for each $k \in K^t$ the set $O_{P,k}^t \subset \{y \in \mathbb{R}^{n-1} : |(y_2, \dots, y_{n-1})| < \rho - t, |y_n| < \sigma\}$ is a (nonempty) open connected set, and $m_{O_{P,k}^t} \neq 0$ is an integer so that $m_{O_{P,k}^t} \in [-m_0, 0)$

whenever $O_{P,k}^t \cap \{y \in \mathbb{R}^{t,n-1} : y_n > 0\} \neq \emptyset$ while $m_{O_{P,k}^t} \in (0, m_0]$ whenever $O_{P,k}^t \cap \{y \in \mathbb{R}^{t,n-1} : y_n < 0\} \neq \emptyset$.

For each $k \in K$ (as in (A.0.4)), we can choose $t \in (-\rho, \rho)$ so that $O_{P,k} \cap \{y \in \mathbb{R}^n : y_1 = t\} \neq \emptyset$ and so that $\langle P, \mathbf{p}_1, t \rangle \neq 0$ exists. Thus $m_{O_{P,k}} = m_{O_{P,\tilde{k}}^t}$ for some $\tilde{k} \in K^t$. We conclude the lemma. \square

The following calculations will be used throughout, and hence we collect them here. In particular, (A.0.8) is instrumental in checking that the proofs of [14] carry over to the more general setting of Theorem 2; see the proof of Lemma 8.4, analogous to Lemma 6.4 of [14]. On the other hand, the fact that (A.0.8) holds in general only with $m \geq 1$ means we can only presently prove Theorem 2 with $m \geq 1$.

Lemma A.0.5. *Let $M \in \mathbb{N}$, $m \in \{0, \dots, M-1\}$, and let $\alpha \in (0, 1]$. Suppose $T \in \mathcal{T} = \mathcal{T}(M, m, \alpha)$ (see Definition 3.3). With $q : \mathbb{R} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ given by*

$$q(t, x) = (x_1, \dots, x_{n-1}, tx_n, tx_{n+1}) \text{ for } (t, x) \in \mathbb{R} \times \mathbb{R}^{n+1},$$

define $Q_T \in \mathcal{R}_n(\mathbb{R}^{n+1})$ by

$$Q_T = q_{\sharp}((\mathbb{E}^1 \llcorner [0, 1]) \times (\partial T \llcorner C_2))$$

(see 4.1.9 of [10]). Then for every $r \in (0, 2)$

$$\begin{aligned} \mathbf{M}(Q_T \llcorner C_r) &\leq (M - m) \kappa_T \varpi_{n-1} r^{n+\alpha} \\ (A.0.6) \quad &\times \left(\frac{\alpha}{2} \right) \left(1 + \frac{\alpha^2 \kappa_T^2}{4} r^{2\alpha} + \frac{\alpha^4 \kappa_T^4}{16} r^{4\alpha} \right)^{\frac{1}{2}}, \\ \mathbf{M}(\mathbf{p}_{\sharp} Q_T \llcorner C_r) &\leq (M - m) \kappa_T \varpi_{n-1} r^{n+\alpha}, \end{aligned}$$

and we as well have

$$\begin{aligned} (A.0.7) \quad \partial Q_T \llcorner C_r &= \partial T \llcorner C_r - (-1)^n (M - m) \mathbb{E}^{n-1} \llcorner C_r \\ (\mathbf{p}_{\sharp} T) \llcorner \bar{B}_r^n &= (M \mathbb{E}^n \llcorner \mathbf{V} + m \mathbb{E}^n \llcorner \mathbf{W} + \mathbf{p}_{\sharp} Q_T) \llcorner \bar{B}_r^n. \end{aligned}$$

If $m \geq 1$, $\sigma \in (0, 1/2)$, and $\kappa_T < \sigma$ (see (3.3.3)), then

$$\begin{aligned} (A.0.8) \quad \mathbf{M}((\mathbf{p}_{\sharp} T - \mathbb{E}^n) \llcorner \{y \in B_{\frac{1}{2}}^n : |y_n| < \sigma\}) \\ = \mathbf{M}(\mathbf{p}_{\sharp} T \llcorner \{y \in B_{\frac{1}{2}}^n : |y_n| < \sigma\}) \\ - \mathbf{M}(\mathbb{E}^n \llcorner \{y \in B_{\frac{1}{2}}^n : |y_n| < \sigma\}). \end{aligned}$$

Proof. First, we compute by 4.1.9 of [10] (see as well the proof of 26.23 of [26]) and (3.3.2),(3.3.3) (as in the end of the proof of Lemma 5.2)

$$\begin{aligned}
\mathbf{M}(Q_T \mathbf{L} C_r) &\leq \int \sqrt{\mathbf{q}_0^2 + \mathbf{q}_1^2} d\mu_{\partial T \mathbf{L} C_r} \\
&\leq \left(\frac{\alpha}{2}\right) \kappa_T r^{1+\alpha} \mu_{\partial T}(C_r) \\
&\leq (M-m) \kappa_T \varpi_{n-1} r^{n+\alpha} \\
&\quad \times \left(\frac{\alpha}{2}\right) \left(1 + \frac{\alpha^2 \kappa_T^2}{4} r^{2\alpha} + \frac{\alpha^4 \kappa_T^4}{16} r^{4\alpha}\right)^{\frac{1}{2}}.
\end{aligned}$$

By a similar calculation for $\mathbf{M}(\mathbf{p}_\# Q_T \mathbf{L} C_r)$, we conclude (A.0.6).

Second, for any $r \in (0, 2)$ we have by 4.1.8-9 of [10] (see as well 26.22 of [26], the homotopy formula) the first identity in (A.0.7) and

$$\partial(\mathbf{p}_\# T - M \mathbb{E}^n \mathbf{L} \mathbf{V} - m \mathbb{E}^n \mathbf{L} \mathbf{W} - \mathbf{p}_\# Q_T) \mathbf{L} \bar{B}_r^n = 0,$$

recalling that $\sum_{\ell=1}^N m_\ell = M - m$ by (3.3.2). This proves the second identity in (A.0.7), by the constancy theorem and (3.3.4).

Third, suppose $m \geq 1$ and with $\sigma \in (0, 1/2)$ assume $\kappa_T < \sigma$. Let $P = (\mathbf{p}_\# Q_T) \mathbf{L} B_1^n$. If $P = 0$, then (A.0.8) readily follows from (A.0.7) (since $1 \leq m \leq (M-1)$). Now suppose $P \neq 0$, then we first note that (3.3.3),(3.3.4), and (A.0.7) imply

$$\begin{aligned}
\text{spt } P \cap \{y \in \mathbb{R}^n : |(y_1, \dots, y_{n-1})| < 1/2, |y_n| < 1\} \\
\subset \{y \in \mathbb{R}^n : |(y_1, \dots, y_{n-1})| < 1/2, |y_n| < \sigma\}.
\end{aligned}$$

This together with (3.3.2),(A.0.7) means we apply Lemma A.0.1 with $\rho = 1/2$ and $m_0 = M - m$ to conclude

$$P \mathbf{L} \{y \in \mathbb{R}^n : |(y_1, \dots, y_{n-1})| < 1/2, |y_n| < 1\} = \sum_{k \in K} m_{O_{P,k}} \mathbb{E}^n \mathbf{L} O_{P,k},$$

where we recall $m_{O_{P,k}} \in [-(M-m), 0)$ if $O_{P,k} \cap \{y \in \mathbb{R}^n : y_n > 0\} \neq \emptyset$ while

$m_{O_{P,k}} \in (0, (M - m)]$ if $O_{P,k} \cap \{y \in \mathbb{R}^n : y_n < 0\} \neq \emptyset$. By further writing

$$\begin{aligned}
P\mathbf{L}\{y \in \mathbb{R}^n : |(y_1, \dots, y_{n-1})| < 1/2, |y_n| < 1\} \\
&= \sum_{\{k \in K : O_{P,k} \cap \mathbf{W} = \emptyset\}} m_{O_{P,k}} \mathbb{E}^n \mathbf{L} O_{P,k} \\
&\quad + \sum_{\{k \in K : O_{P,k} \cap \mathbf{W} \neq \emptyset\}} m_{O_{P,k}} \mathbb{E}^n \mathbf{L} (O_{P,k} \cap \mathbf{V}) \\
&\quad + \sum_{\{k \in K : O_{P,k} \cap \mathbf{W} \neq \emptyset\}} m_{O_{P,k}} \mathbb{E}^n \mathbf{L} (O_{P,k} \cap \mathbf{W}),
\end{aligned}$$

then using (A.0.7) we can compute each side of (A.0.8) in terms of $M, m, \{m_{O_{P,k}}\}_{k \in K}$ in order to verify equality (for this, $m \geq 1$ is needed). \square

The following calculation, (A.0.10), is used in the proof of Lemma 6.1. We prove it here for the sake of cleaner exposition.

Lemma A.0.9. *Let $M \in \mathbb{N}$, $m \in \{0, \dots, M - 1\}$, $\alpha \in (0, 1]$, and suppose $T \in \mathcal{T} = \mathcal{T}(M, m, \alpha)$ (see Definition 3.3). Also suppose $\tau \in (0, 1)$ and that $\kappa_T \leq 4 \cdot 3^n (1 + M\varpi_n)\tau^2$ (see (3.3.3)). Let $\phi \in C^1(\mathbb{R}^n; [0, 1])$ satisfy*

$$\begin{aligned}
\phi(y) &= 0 && \text{if } |y| \leq 1 \\
0 < \phi(y) &< 1 && \text{if } 1 < |y| < 1 + \tau \\
\phi(y) &= 1 && \text{if } 1 + \tau \leq |y| \\
|D\phi(y)| &\leq 3/\tau && \text{for all } y \in \mathbb{R}^n,
\end{aligned}$$

and define $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ and $h : \mathbb{R} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ by

$$F(x) = (\mathbf{p}(x), f(\mathbf{p}(x))x_{n+1}) \text{ and } h(t, x) = (1 - t)F(x) + tx.$$

Then $R_T = h_{\#}([0, 1] \times \partial T)$ satisfies the mass bound

$$(A.0.10) \quad \mathbf{M}(R_T) \leq \left(\frac{\sqrt{21}}{8} + 2^{\frac{9n-7}{2}} 3^{n^2 - \frac{1}{2}} \right) (M - m)\varpi_{n-1}(1 + M\varpi_n)^{n-1} \kappa_T.$$

Proof. We compute using 4.1.9 of [10] (see also the end of section 5.2),

$$\begin{aligned}
\mathbf{M}(R_T) &\leq \int (1 - \phi) |\mathbf{q}_1| \max \left\{ 1, (1 + \phi^2 + \mathbf{q}_1^2 |D\phi|^2)^{\frac{n-1}{2}} \right\} d\mu_{\partial T} \\
&\leq \left(\frac{\alpha}{2} \kappa_T \right) \mu_{\partial T}(C_1) \\
&\quad + \left(\frac{\alpha}{2} \kappa_T \right) (1 + \tau)^{1+\alpha} \left(2 + \left(\frac{\alpha^2}{4} \right) \kappa_T^2 (1 + \tau)^{2+2\alpha} \left(\frac{9}{\tau^2} \right) \right)^{\frac{n-1}{2}} \\
&\quad \times \mu_{\partial T}(C_{1+\tau} \setminus C_1) \\
&\leq \frac{1}{2} \left(\mu_{\partial T}(C_1) + 2^2 \left(2 + \frac{36\kappa_T^2}{\tau^2} \right)^{\frac{n-1}{2}} \mu_{\partial T}(C_{1+\tau} \setminus C_1) \right) \kappa_T \\
&\leq \frac{1}{2} \left(\mu_{\partial T}(C_1) + 2^2 (2 + 2^6 \cdot 3^{2n+2} (1 + M\varpi_n)^2)^{\frac{n-1}{2}} \mu_{\partial T}(C_{1+\tau} \setminus C_1) \right) \kappa_T \\
&\leq \frac{1}{2} \left(\mu_{\partial T}(C_1) + 2^{\frac{7n-3}{2}} \cdot 3^{n^2-1} (1 + M\varpi_n)^{n-1} \mu_{\partial T}(C_{1+\tau} \setminus C_1) \right) \kappa_T \\
&\leq \frac{1}{2} \left(\frac{\sqrt{21}}{4} + 2^{\frac{9n-5}{2}} 3^{n^2-\frac{1}{2}} \right) (M - m) \varpi_{n-1} (1 + M\varpi_n)^{n-1} \kappa_T.
\end{aligned}$$

□

We will also need the following lemma for the proof of Lemma 8.1. Again, we give it here for the sake of cleaner exposition.

Lemma A.0.11. *Let $M \in \mathbb{N}$, $m \in \{0, \dots, M-1\}$, $\alpha \in (0, 1]$, and suppose $T \in \mathcal{T} = \mathcal{T}(M, m, \alpha)$ (see Definition 3.3). Also suppose there is a $\sigma \in (0, 1)$ so that*

$$(A.0.12) \quad \mathbf{E}_C(T, 1) + \kappa_T < \frac{\left(\frac{2}{3}\right)^n \varpi_n}{(1 + (M - m) \varpi_{n-1} + c_4)} \sigma^{n+1}$$

with c_4 as in (4.2.4), (4.2.5), and

$$(A.0.13) \quad \partial(T \llcorner B_{1-\frac{\sigma}{6}}) \llcorner C_{1-\frac{\sigma}{3}} = (\partial T) \llcorner C_{1-\frac{\sigma}{3}}.$$

Then

$$\mathbf{p}_\#((T \llcorner B_{1-\frac{\sigma}{6}}) \llcorner C_{1-\frac{\sigma}{3}}) = \mathbf{p}_\#(T \llcorner C_{1-\frac{\sigma}{3}}).$$

Proof. First, we compute using (4.2.5) and (3.3.1)

$$\left(\frac{M+m}{2}\right) \varpi_n - \frac{\mu_T(\bar{B}_{1-\frac{\sigma}{3}})}{\left(1-\frac{\sigma}{3}\right)^n} = -\mathbf{E}_S\left(T, 1-\frac{\sigma}{3}\right) \leq c_4 \kappa_T.$$

This together with Lemma 26.25 of [26], (3.2.1), and (A.0.6),(A.0.7) gives

$$\begin{aligned} & \mathbf{M}\left(\mathbf{p}_\#(T \llcorner C_{1-\frac{\sigma}{3}} \setminus \bar{B}_{1-\frac{\sigma}{3}})\right) \\ & \leq \mu_T(C_{1-\frac{\sigma}{3}} \setminus \bar{B}_{1-\frac{\sigma}{3}}) \\ & = \left(1-\frac{\sigma}{3}\right)^n \mathbf{E}_C\left(T, 1-\frac{\sigma}{3}\right) + \mu_{\mathbf{p}_\#T}(C_{1-\frac{\sigma}{3}}) - \mu_T(\bar{B}_{1-\frac{\sigma}{3}}) \\ & \leq \mathbf{E}_C(T, 1) + \mu_{\mathbf{p}_\#T}(C_{1-\frac{\sigma}{3}}) - \mu_T(\bar{B}_{1-\frac{\sigma}{3}}) \\ & \leq \mathbf{E}_C(T, 1) + \left(\frac{M+m}{2}\right) \varpi_n \left(1-\frac{\sigma}{3}\right)^n \\ & \quad + (M-m) \kappa_T \varpi_{n-1} \left(1-\frac{\sigma}{3}\right)^{n+\alpha} - \mu_T(\bar{B}_{1-\frac{\sigma}{3}}) \\ & \leq \mathbf{E}_C(T, 1) + (M-m) \kappa_T \varpi_{n-1} \left(1-\frac{\sigma}{3}\right)^{n+\alpha} + \left(1-\frac{\sigma}{3}\right)^n c_4 \kappa_T \\ & \leq (1 + (M-m) \varpi_{n-1} + c_4) (\mathbf{E}_C(T, 1) + \kappa_T), \end{aligned}$$

recalling as well $\sigma \in (0, 1)$. Combining with (A.0.12) gives

$$\mathbf{M}\left(\mathbf{p}_\#(T \llcorner C_{1-\frac{\sigma}{3}} \setminus \bar{B}_{1-\frac{\sigma}{3}})\right) \leq \left(\frac{2}{3}\right)^n \varpi_n \sigma^{n+1}.$$

On the other hand, the constancy theorem (see Theorem 26.27 of [26]) and (A.0.13) imply

$$\mathbf{p}_\#(T \llcorner C_{1-\frac{\sigma}{3}} \setminus \bar{B}_{1-\frac{\sigma}{3}}) = \tilde{m} \mathbb{E}^n \llcorner \bar{B}_{1-\frac{\sigma}{3}}^n$$

for some integer \tilde{m} . The above estimate implies we need

$$|\tilde{m}| \left(\frac{2}{3}\right)^n \varpi_n \leq |\tilde{m}| \left(1-\frac{\sigma}{3}\right)^n \varpi_n \leq \left(\frac{2}{3}\right)^n \varpi_n \sigma^{n+1}.$$

Since $\sigma \in (0, 1)$ we conclude $\tilde{m} = 0$, from which the lemma follows. \square

References

- [1] Allard, W.K.: On the first variation of a varifold. Ann. Math. 95, 417-491 (1972)

- [2] Allard, W.K.: On the first variation of a varifold-boundary behavior. *Ann. Math.* 101, 418-446 (1975)
- [3] Bourni, T.: Allard type boundary regularity for varifolds with $C^{1,\alpha}$ boundary. preprint (2010) <http://arxiv.org/abs/1008.4728>
- [4] Brothers, J.E.: Existence and Structure of tangent cones at the boundary of an area-minimizing integral current. *Indiana U. Math. J.* 26, 1027-1044, (1977)
- [5] Barbosa J.L., DoCarmo, M.: On the size of a stable minimal surface in \mathbb{R}^3 . *Amer. J. Math.*, 98 515-528 (1976)
- [6] Bombieri, E., DeGiorgi, E., Miranda, M.: Una maggionazione a priori relative alle ipersuperfici minimali non-parametriche. *Arch. Rat. Mech. Anal.*, 32, 255-267, (1969)
- [7] Duzaar, F., Steffen, K.: Boundary regularity for minimizing currents with prescribed mean curvature. *Calc. Var.* 1, 355-406 (1993)
- [8] Duzaar, F., Steffen, K.: Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals. *J. Reine Angew. Math.* 546, 73-138 (2002)
- [9] Ecker, K.: Area-minimizing integral currents with movable boundary parts of prescribed mass. *Ann. I. H. Poincaré-An.* 6, 261-293 (1989)
- [10] Federer, J.: *Geometric Measure Theory*. Springer-Verlag New York Inc, New York, (1969)
- [11] Finn, R., Gilbarg, T.: Subsonic flows. *Commun. Pur. Appl. Math.* 10, 23-63 (1957)
- [12] Gilbarg, D., Trudinger, N.S.: *Elliptic partial differential equations of second order*. Springer-Verlag Berlin, Heidelberg (1983)
- [13] Hardt, R.: On boundary regularity for integral currents or flat chains modulo two minimizing the integral of an elliptic integrand. *Comm. in P.D.E.* 2, 1163-1232 (1977)
- [14] Hardt, B., Simon, L.: Boundary regularity and embedded solutions for the oriented Plateau problem. *Ann. Math.* 110, 439-486, (1979)

- [15] Hoffman, D.A.: Surfaces of constant mean curvature in manifolds of constant curvature. *J. Differ. Geom.* 8, 161-176 (1973)
- [16] Jenkins, H., Serrin, J.: Variational problems of minimal surface type I. *Arch. Rat. Mech. Anal.* 35, 47-82, (1963)
- [17] Morrey Jr., C.B.: Multiple integrals in the calculus of variations. Springer-Verlag, Berlin-Heidelberg-New York, (1966)
- [18] Moser, J.: A new proof of DeGiorgi's theorem concerning the regularity problem for elliptic differential equations. *Comm. Pure Appl. Math.* 13, 457-463, (1960)
- [19] Michael, J., Simon, L.: Sobolev and mean value inequalities on generalized submanifolds of \mathbb{R}^n . *Comm. Pur. Appl. Math.* 26, 361-379, (1973)
- [20] Rosales, L.: The geometric structure of solutions to the two-valued minimal surface equation. 39, 59-84, (2010)
- [21] Rosales, L.: The c -isoperimetric mass of currents and the c -Plateau problem. *J. Geom. Anal.* 25, 471-511, (2015)
- [22] Rosales, L.: Partial boundary regularity for co-dimension one area-minimizing currents at immersed $C^{1,\alpha}$ tangential boundary points. (preprint 2015) arXiv:1508.04229v2 [math.DG]
- [23] Rosales, L.: Two-dimensional solutions to the c -Plateau problem in \mathbb{R}^3 . 50, 129-163, (2016)
- [24] Rosales, L.: Co-dimension one area minimizing currents with $C^{1,\alpha}$ tangentially immersed boundary. (preprint 2016) arXiv:1603.08568 [math.DG]
- [25] Simon, L.: Interior gradient bounds for non-uniformly elliptic equations. *Indiana Univ. Math. J.*, 25, 821-855, (1976)
- [26] Simon, L.: Lectures on Geometric Measure Theory. Centre for Mathematical Analysis, Australian National University, Australia, (1984)
- [27] Simon, L., Wickramasekera, N.: Stable branched minimal immersions with prescribed boundary. *J. Differ. Geom.* 75, 143-173, (2007)

- [28] Treves, F.: Basic Linear Partial Differential Equations. Academic Press, New San Francisco, London (1975)
- [29] Trudinger, N.: A new proof of the interior gradient bound for the minimal surface equation in n -dimensions. Proc. Nath. Acad. Sci. U.S.A. 69, 821-823, (1972)
- [30] White, B.: Regularity of area-minimizing hypersurfaces at boundaries with multiplicity. Ann. Math. Studies, 103, 293-301, (1983)
- [31] Wickramasekera, N.: A general regularity theory for stable codimension 1 integral varifolds. Ann. Math., 179, 843-1007, (2014)